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The q-State Potts Model in the Standard Pirogov–Sinai Theory: Surface Tensions and Wilson Loops

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The q-state Potts model (both scalar and gauge versions) is rewritten, with the help of the duality transformation, into a form of the Pirogov–Sinai theory with noninteracting contours that can be controlled by cluster expansions once q is large enough. This is then used in a new proof of the existence of a unique transition (inverse) temperature β_t, where the mean internal energy is discontinuous. Moreover, we prove for the scalar model (again for q large enough) that there are discontinuities at β_t of the magnetization and of the mass gap, with the magnetization vanishing below β_t and the mass gap vanishing above β_t. We also show that the surface tensions between ordered stable phases are strictly positive up to β_t, and the surface tension between an ordered phase and the disordered one is strictly positive at β_t. For the three-dimensional gauge model, the Wilson parameter exhibits a direct transition from an area law decay (quark confinement) to a perimeter law decay (deconfinement).

KEY WORDS: Potts model; phase transition; surface tension; string tension; Wilson loop; duality transformation; Pirogov–Sinai theory; combinatorial topology.

1. INTRODUCTION AND RESULTS

1.1. Introduction

The Potts model was introduced in 1952[1] as a generalization of the Ising model by enlarging the number of values taken by the "spin" on each site

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from two to an arbitrary integer value q. What makes the Potts model interesting is that its simple structure permits a rather precise analysis of its phase diagram. Moreover, it exhibits a temperature-driven first-order phase transition at some point where $q+1$ phases coexist. A proof of the magnetization discontinuity in dimension 2 for $q>4$ was given in ref. 2. In dimension $d \geqslant 2$ one sees by using the usual Peierls argument that there are q translation-invariant Gibbs states at low temperatures. These q states coexist at the transition temperature with yet another state, the disorder one. The transition can be interpreted in terms of the energy entropy fighting,[3] the q ordered states correspond to perturbations of the ground states of the Hamiltonian, while the $(q+1)$th one corresponds to a purely entropic state associated with the set of configurations where all nearest neighbor spins are different.

A proof of the existence of a first-order transition (for q large) is presented in ref. 4, where also the gauge formulation introduced by Kogut[5] according to the Wilson formulation of field theory on the lattice[6] is analyzed. The proof is based on the use of the reflection positivity.[7] It is, however, natural and interesting to try to understand the transition in terms of the Pirogov–Sinai theory.[8] In ref. 9 a generalization of that theory was given with the notion of ground states replaced by measures over suitable subsets of the configuration space called restricted ensembles. In the case of the Potts model the restricted ensemble corresponding to the disordered state is the set of configurations where all nearest neighbor spins are different. In this theory contour models describing pure phases involve interacting contours and a good control on the decay of correlations in restricted ensembles is needed (diluteness hypothesis). A generalization of the Pirogov–Sinai theory, where only standard contour model are used, is given in ref. 10, where a version of the diluteness hypothesis is also needed. Interacting contour models have been also introduced independently in ref. 11 and used in ref. 12 to analyze the set of translation-invariant Gibbs states.

Another approach, which consists in transforming the $(q+1)$th disordered phase into an ordered one, was initiated in ref. 13. It is based on an observation that the duality transformation turns a model with the free boundary conditions into a dual model with the ordered boundary conditions and a prefactor that captures the entropy of the disordered restricted ensemble. It turns out that this observation is useful also for Potts models that are not self-dual, e.g., the three-dimensional scalar model that is transformed by duality into the three-dimensional gauge model (let us recall that the two-dimensional scalar model and the four-dimensional pure gauge model are self-dual and the duality was used in refs. 13 and 14 as an exact symmetry at the transition (self-dual) temperature to evaluate the probability of suitable contours separating ordered and disordered phases). Namely, it allows us to enclose Potts models, supposing q is large enough, into the framework of the standard Pirogov–Sinai theory with noninteracting contours. It might seem contradictory that a theory with noninteracting contours is obtained; in Section 2.1 we explain in nontechnical terms the main idea of how it is achieved.

Having formulated the models in terms of the Pirogov–Sinai theory, we use it to control the behavior around the transition temperature and to derive few new results (and rederive some old ones) concerning the scalar and the gauge Potts models. Namely, assuming that q is large enough, we prove for the d-dimensional scalar model ($d \geqslant 2$) the existence of a unique first-order transition point β_t where q ordered phases coexist with a disordered one. The transition point β_t is thus that (inverse) temperature at which the (suitably defined) magnetization jumps to zero as β decreases. The mass gap (inverse of the correlation length defined with the free boundary condition) jumps at the same time from zero to a positive value. We also get a control over surface tensions between coexisting phases (two different ordered phases up to β_t and an ordered phase with the disordered one at β_t). For the three-dimensional gauge model we prove the existence of a unique first-order transition point β'_t where the model exhibits a direct transition from an area law decay to a perimeter law decay.

Notice that these results imply the absence of intermediate phase for Potts and gauge Potts models for large q. Such results were previously known for small values of q, namely for the percolation model ($q = 1$) and the Ising model ($q = 2$). The sharpness of the transition for the bond percolation model was first proved in two dimension (15, 16) and more recently in three dimensions.[17] For the plaquette percolation model, the direct transition from an area law decay to a perimeter law decay has been showed in dimension three,[18] provided the transition of the dual three-dimensional bond percolation model is sharp. The absence of intermediate phase is also known for the Ising model in dimension greater than two,[19] and the direct transition from an area law decay to a perimeter law decay is proved for the three-dimensional Ising gauge model.[20] Let us stress that these latter transitions are not first order in the considered dimensions, and thus nonperturbative arguments had to be employed, in contrast to our case, where convergent expansions for q large is used.

The paper is organized as follows. We state our main results in Section 1.2 and explain the strategy on how to enclose Potts models in the framework of the standard Pirogov–Sinai theory in Section 2.1. In Section 2.2 we introduce the duality transformation. To this end, we find it useful to use the cell-complex formalism. For the reader's convenience we summarize it in Appendices A.1 and A.2. In Section 2.3 we define contours and in Section 2.4 we use them to formulate an inductive expression for the partition functions. This is our starting point in Section 2.5, containing our main proposition, which allows us to write partition functions in terms of contours models with "good functions" provided a "generalized Peierls condition" is satisfied. This condition is proved in Appendix B for the scalar models with $d \geqslant 2$ and for the gauige model with $d = 3$. The proof requires some topological considerations that need further topological concepts concerning cell-complexes added in Appendix A.3. Finally, the proofs of the theorems are given in Section 3.

1.2. Results

To introduce the scalar q-state Potts model, we associate with each lattice site $x = \{x^1,..., x^d\}$ of $\mathbb{Z}^d$ a spin σ_x taking values in the set $\{0, 1,..., q-1\}$. Let $\Lambda \subset \mathbb{Z}^d$ be a finite set. With a configuration $\sigma_\Lambda \equiv \{\sigma_x\}$, $x \in \Lambda$, we associate the Hamiltonian

$$H_\Lambda^{\mathrm{bc}}(\sigma_\Lambda) = -\sum_{\langle x; y\rangle \subset \Lambda} \delta_{\sigma_x, \sigma_y} - \sum_{\substack{\langle x; y\rangle: \\ x \in \Lambda;\, y \in \mathbb{Z}^d/\Lambda}} \delta_{\sigma_x, \tilde{\sigma}_y} \tag{1.2.1}$$

where δ is the Kronecker symbol, $\delta_{\sigma,\sigma'} = 1$ if $\sigma = \sigma'$ and zero otherwise, the two sums are over nearest neighbor pairs, and $\tilde{\sigma}$ is fixed outside Λ and represents the boundary conditions (bc). In particular, we shall consider in the following:

(i) The free (denoted f) boundary conditions: the second term in the rhs of (1.2.1) is omitted.

(ii) The ordered (α) boundary conditions: all the spins $\tilde{\sigma}_y$ take the value α.

(iii) The mixed (α_1, α_2) boundary conditions: $\tilde{\sigma}_y = \alpha_1$ if $y^1 \geqslant 0$, $\tilde{\sigma}_y = \alpha_2$ if $y^1 < 0$.

(iv) The mixed (α, f) boundary conditions: $\tilde{\sigma}_y = \alpha$ if $y^1 \geqslant 0$ and the terms from the second sum are omitted for $y^1 < 0$.

Gibbs measures in a volume Λ under certain boundary conditions (bc) at inverse temperature β are probability measures that assign to a configuration σ_Λ the probability

$$\mu_{\Lambda;\beta}^{\mathrm{bc}}(\sigma_\Lambda) = [Z_\Lambda^{\mathrm{bc}}(\beta)]^{-1}\, e^{-\beta H_\Lambda^{\mathrm{bc}}(\sigma_\Lambda)} \tag{1.2.2}$$

with $Z_\Lambda^{\mathrm{bc}}(\beta)$ the corresponding partition function.

We use $\langle g\rangle_\Lambda^{\mathrm{bc}}(\beta)$ to denote the expectation of a measurable function g with respect to the Gibbs measures (1.2.2) in a volume Λ,

$$\langle g\rangle_\Lambda^{\mathrm{bc}}(\beta) = \sum_{\sigma_\Lambda \in \Omega_\Lambda} g(\sigma_\Lambda)\, \mu_{\Lambda;\beta}^{\mathrm{bc}}(\sigma_\Lambda)$$

where Ω_Λ is the set of configurations in Λ. Here $\langle \cdot \rangle^{\mathrm{bc}}(\beta)$ denotes the corresponding infinite-volume limit (satisfying DLR equations). The free energy of the system is defined by

$$f(\beta H) = \lim_{\Lambda \uparrow \mathbb{Z}^d} \frac{1}{|\Lambda|} \log Z_\Lambda^{\mathrm{bc}}(\beta)$$

where $|\Lambda|$ denotes the number of points in Λ and the limit is taken in the van Hove sense.

Our first results are contained in the following theorems.

Theorem 1.1. Whenever $d \geqslant 2$ and q is large enough, there exists a unique first-order transition point $\beta_t(q)$ where the derivative of the free energy with respect to β is discontinuous. More explicitly, if x and y are nearest neighbors, one has

(a) $\langle \delta_{\sigma_x, \sigma_y} \rangle^\alpha(\beta_t) > 1/2 \qquad \alpha \in \{0,..., q-1\}$

(b) $\langle \delta_{\sigma_x, \sigma_y} \rangle^{\mathrm{f}}(\beta_t) < 1/2$

Introducing the magnetization $M(\beta) = [1/(q-1)] \langle q\delta_{\sigma_x, \alpha} - 1 \rangle^\alpha(\beta)$ and the mass gap, defined, say, as the inverse correlation length along the axis $(1, 0,..., 0)$,

$$m(\beta) = -\operatorname*{Lim}_{x \uparrow \infty} \frac{1}{x} \log \left[\frac{1}{q-1} \langle q\delta_{\sigma(0,...,0),\, \sigma(x,0,...,0)} - 1 \rangle^{\mathrm{f}}(\beta) \right]$$

we prove that both have a discontinuity at β_t.

Theorem 1.2. Supposing $d \geqslant 2$ and q is large enough, one has

(a) $M(\beta) = 0$ for $\beta < \beta_t$ and $M(\beta) > 0$ for $\beta \geqslant \beta_t$

(b) $m(\beta) > 0$ for $\beta \leqslant \beta_t$ and $m(\beta) = 0$ for $\beta > \beta_t$

Once one knows that several phases coexist, an interesting problem is the existence or nonexistence of a surface tension between these phases. In thermodynamic systems the bulk free energy of two coexisting phases equals $F(1) + F(2) + |S| \tau^{1,2}$, where $F(1)$ and $F(2)$ are the free energies of the pure phases (1) and (2), respectively, and $|S|$ is the area of the surface separating (1) and (2). The factor $\tau^{1,2}$ is called *a surface tension* (interfacial free energy). In particular, for a lattice model with spins taking values in a discrete finite group, one usually enforces two phases (1) and (2) to coexist (see refs. 21 and 22 for a review and results) by considering asymmetric boundary conditions around a finite rectangular box $\Lambda \subset \mathbb{Z}^d$. More precisely, this situation will be generated if one imposes the boundary condition (1) [which favors the pure phase (1)] around the top half of the box Λ and the boundary conditions (2) [which favor the pure phase (2)] around the bottom half of Λ. In the following we will be interested in two cases of phase coexistence and of surface tensions:

1. The surface tension $\tau^{\alpha_1,\alpha_2}(\beta)$ between two ordered phases α_1 and α_2.

2. The surface tension $\tau^{\alpha,\mathrm{f}}(\beta)$ between an ordered phase α and the "disordered" of free phase.

A definition of $\tau^{\alpha,\mathrm{f}}(\beta)$ was suggested in ref. 23 and studied in ref. 13. Let

$$\Lambda \equiv \Lambda_{L,M} = \{x = (x^1,..., x^d) \in \mathbb{Z}^d \mid 0 \leqslant x^1 \cdots x^{d-1} \leqslant L;\ -M-1 \leqslant x^d \leqslant M\}$$

The surface tensions $\tau^{\alpha_1,\alpha_2}(\beta)$ and $\tau^{\alpha,\mathrm{f}}(\beta\ )$, supposing the limit exists, are defined by

$$\tau^{\alpha_1,\alpha_2}(\beta) = -\lim_{L\uparrow\infty;M\uparrow\infty} \frac{1}{L^{d-1}} \log \frac{Z_\Lambda^{\alpha_1;\alpha_2}(\beta)}{[Z_\Lambda^{\alpha_1}(\beta)\, Z_\Lambda^{\alpha_2}(\beta)]^{1/2}}$$

$$\tau^{\alpha,\mathrm{f}}(\beta) = -\lim_{L\uparrow\infty;M\uparrow\infty} \frac{1}{L^{d-1}} \log \frac{Z_\Lambda^{\alpha;\mathrm{f}}(\beta)}{[Z_\Lambda^{\alpha}(\beta)\, Z_\Lambda^{\mathrm{f}}(\beta)]^{1/2}}$$

Theorem 1.3. Supposing $d \geqslant 2$ and q is large, one has (a) the surface tension $\tau^{\alpha_1,\alpha_2}(\beta)$ between two ordered phases is strictly positive if $\beta \geqslant \beta_t$, and (b) the surface tension $\tau^{\alpha,\mathrm{f}}(\beta)$ is strictly positive at the transition point $\beta_t(q)$.

We now turn to the gauge model. In this case the random variables σ_l are attached to the links $l \equiv \langle x, y\rangle$ of the lattice, take their valules in the set $\{0, 1,..., q-1\}$, and satisfy the condition $\sigma_{\langle x,y\rangle} + \sigma_{\langle y,x\rangle} = 0$. For every plaquette p (elementary square) we let $\sigma(p)$ denote as usual the sum mod(q) of the σ_l over the four links of the plaquette p,

$$\sigma(p) = \sigma_{\langle x,y\rangle} + \sigma_{\langle y,z\rangle} + \sigma_{\langle z,t\rangle} + \sigma_{\langle t,x\rangle}$$

Let L denote the set of links of $\mathbb{Z}^d$ and let now Λ be a finite subset of L, $\Lambda \subset L$. The Hamiltonian of a configuration $\sigma_\Lambda = \{\sigma_l\}$, $l \in L$, is

$$H_\Lambda^{\mathrm{bc}}(\sigma_\Lambda) = -\sum_{p\subset\Lambda} \delta_{\sigma(p),0} - \sum_{\substack{p\cap\Lambda\neq\varnothing\\ p\cap\Lambda^c\neq\varnothing}} \delta_{\sigma(p),0}$$

The variables σ_l, $l \in \Lambda^c \equiv L\backslash\Lambda$, in the second sum are fixed and represent the b.c., and (i) we call them the closed (0) b.c. if $\sigma_l = 0$ for $l \in \Lambda^c$, and (ii) the free b.c. if the second sum is omitted.

In analogy with the scalar model, we introduce Gibbs measures, the partition function, and expectations. Notice that both the Hamiltonian and the measure are invariant under local gauge transformations

$$\sigma_{\langle x, y \rangle} \to \sigma'_{\langle x, y \rangle} = \sigma_{\langle x, y \rangle} + \tau_x - \tau_y$$

where τ_x is a random variable attached to the lattice site x.

Theorem 1.4. If $d=3$ and q is large enough, there exists a first-order transition point $\beta'_t(q)$ where the derivative of the free energy with respect to β is discontinuous:

(a) $\langle \delta_{\sigma(p),00} \rangle^0 (\beta'_t) > 1/2$
(b) $\langle \delta_{\sigma(p),0} \rangle^{\mathrm{f}} (\beta'_t) < 1/2$

We shall now consider the Wilson parameter. To introduce it, we attach to a loop $\mathscr{L}$ of size $L \cdot T$ ($\mathscr{L} \equiv \{\langle x_1 x_2 \rangle, \langle x_2 x_3 \rangle, \langle x_3 x_4 \rangle, ..., \langle x_n x_1 \rangle\}$) the variable

$$\sigma(\mathscr{L}) = \sum_{l \in L}^{\operatorname{mod}(q)} \sigma_l$$

The Wilson parameter is the expectation of $[1/(q-1)][q\delta_{\sigma(\mathscr{L}),0} - 1]$, while the Wilson string tension $s(\beta)$ is defined[24] by

$$s(\beta) = -\lim_{L \cdot T \uparrow \infty} \frac{1}{L \cdot T} \log \left[\frac{1}{q-1} \langle q\delta_{\sigma(\mathscr{L}),0} - 1 \rangle^{\mathrm{f}} (\beta) \right]$$

It turns out that the Wilson parameter exhibits a direct transition from a regime of area law decay to a regime of perimeter law decay. Moreover, the Wilson string tension is discontinuous at β'_t.

Theorem 1.5. If $d=3$ and q is large enough, there exist constants k and $k' > 0$ such that one has:

(a) $$\frac{1}{q-1} \langle q\delta_{\sigma(\mathscr{L}),0} - 1 \rangle^{\mathrm{f}} (\beta) \leqslant e^{-kLT} \qquad \text{if} \quad \beta \leqslant \beta'_t$$

(b) $$\frac{1}{q-1} \langle q\delta_{\sigma(\mathscr{L}),0} - 1 \rangle^{\mathrm{f}} (\beta) \geqslant e^{-k'(L+T)} \qquad \text{if} \quad \beta > \beta'_t$$

(c) $$\frac{1}{q-1} \langle q\delta_{\sigma(\mathscr{L}),0} - 1 \rangle^{0} (\beta) \geqslant e^{-k'(L+T)} \qquad \text{if} \quad \beta \geqslant \beta'_t$$

2. FORMULATION IN TERM OF PIROGOV–SINAI THEORY

2.1. Introduction

To get a control over the bahavior of the model around the transition temperature we shall use the Pirogov–Sinai theory. The problem with its implementation stems from the fact that, although the q ordered phases of the Potts model are expresed as perturbations of the ground states as used in standard Pirogov–Sinai theory, the $(q+1)$th disordered phase coexisting at transition temperature needs for its description a generalization of the notion of ground states. Namely, one takes for the "disordered ground state" the corresponding restricted ensemble,[9] i.e., the collection of entirely disordered configurations. The aim of Pirogov–Sinai theory now is to describe the disordered phase as the sea of the disordered ground state perturbed by small islands of order. This is achieved by expressing the probability of a given set of external contours (the boundaries of islands) as the probability of the same set for suitably chosen contour models. However, this is not possible in a direct way. While the entropy of the disordered ground state in the exterior of contours, and thus also their probability, depends slightly on their mutual positions, it does not for any standard contour models.

Two generalizations of Pirogov–Sinai theory were suggested to overcome this obstacle. The method in ref. 9 is based on the introduction of "contour models with interactions." In ref. 10 a standard contour model is used for every fixed disordered configuration on the exterior of contours and the final probability of contours is then expressed in terms of means of those contour models with respect to restricted ensemble. A good control of the decay of correlations in restricted ensemble (diluteness hypothesis) is crucial for both approaches.

Here we suggest an implementation of standard Pirogov–Sinai theory. This seems contradictory in the light of what was said above. Therefore, we shall first explain the idea of how it is achieved. All the technical details and full definitions are to be found in Sections 2.3 and 2.4. Our approach is based on the observation that a model in a finite volume with free boundary conditions transforms by duality into the corresponding dual model with spins ordered on the boundary. On the other hand, the free boundary conditions correspond to the disordered phase and the dual transformation brings a prefactor which actually expresses the entropy of the disordered "restricted ensemble" (this idea was introduced in ref. 13); all this suggests that whenever in the process of describing the model in terms of contours one meets the disordered phase, one should replace it with an ordered phase of the dual model. Indeed it turns out that one may follow consistently these ideas. At the end we get a model with contours separating regions with ordered configurations of the original model and of its dual transform. We then apply the standard Pirogov–Sinai theory to this model, obtaining an expression in terms of two standard models with parameters, one for the ordered phase of the original model and one for the ordered phase of the dual model. The latter yields relevant information about the disordered phase of the original model. The transition temperature is then obtained as the unique temperature for which parameters of both models vanish. We found it useful to introduce these contour functionals in a constructive way using the inductive proocedure proposed in ref. 25.

To prove the τ-functionality, as one usually calls the condition yielding a good control of resulting contour models, one uses a generalized Peierls condition which expresses the fact that one pays, either by entropy or by energy, for creating an additional contour. It is natural (and based on an experience from ref. 4) to expect that such a condition is valid for q large at all temperatures. It is so for our version of the Peierls condition, a feature that our approach shares with ref. 10.

2.2. Potts Model and Duality Transformation

Hereafter we shall use the cell complex formalism, which is very convenient when dealing with topological problems arising in calculations with the help of duality, which is crucial in our approach. For the reader's convenience this formalism is summarized in Appendix A.

Let G be an Abelian group and K a cell complex (hereafter we shall only consider a-complexes); a G-valued $(p-1)$-chain $\sigma \in C^{p-1}(K, G)$ on a complex K may be interpreted as a configuration of a lattice model; the case $p=1$ corresponds to a scalar model with spins taking values in G and attached to lattice sites, while $p=2$ corresponds to a gauge model. Considering in particular $\mathbb{Z}_q$-valued chains (we shall represent $\mathbb{Z}_q$ as a set of integers $\{0, 1, 2,..., q-1\}$ with summation modulo q), we may introduce generalized Potts models on a finite cell complex K with partition functions at an inverse temperature β defined by[4]

$$\mathbf{Z}(K, \beta, p) = \sum_{\sigma \in C^{p-1}(K)} e^{-\beta H_K^p(d\sigma)} \tag{2.2.1}$$

$$H_K^p(\varphi^p) = - \sum_{s^p \in K}^{(+)} \delta[\varphi^p(s^p)] \qquad \text{for} \quad \varphi^p \in C^p(K) \tag{2.2.2}$$

[4] Since we shall only consider $\mathbb{Z}_q$-valued chains, to simplify the notations we drop hereafter the corresponding specification and denote $C^p(K)$ instead of $C^p(K, \mathbb{Z}_q)$, and analogously for its subgroups.

where $\delta[\sigma]=\delta_{\sigma,0}$ and the sum in (2.2.2) is only over positively oriented cells. Let us remark that (2.2.2) is used in (2.2.1) with $\varphi^p = d\sigma^{p-1}$, where $d\sigma^{p-1}(s^p)=\sigma^{p-1}(\partial s^p)$. Here the boundary operator is restricted to the complex K (typically a cell subcomplex of the cell complex associated with $\mathbb{Z}^d$ and denoted by $\mathbb{L}$); this is actually a way of introducing certain boundary conditions. In particular, if K is closed (resp. open) $\mathbf{Z}(K, \beta, p)$ is a partition function with the so-called free (resp. ordered "0") boundary condition.

Remark. The sum over σ^{p-1} in (2.2.1) is redundant due to the invariance of the Hamiltonian. Namely, $H^p(d\sigma^{p-1})$ is constant on the group of $\mathbb{Z}_q$-valued cocycles $Z^{p-1}(K)$ and may be expressed as a function on the $\mathbb{Z}_q$-valued coboundary group $B^p(K)$. It will be useful to introduce the gauge fixed (g.f.) partition function (the term being justified by the case $p=2$) by

$$\mathbf{Z}^{\mathrm{g.f.}}(K, \beta, p) = \sum_{b \in B^p(K)} e^{-\beta H^p_K(b)} \tag{2.2.3}$$

Denoting by $|G|$ the cardinality of G, we immediately get

$$\mathbf{Z}(K, \beta, p) = |Z^{p-1}(K)|\ \mathbf{Z}^{\mathrm{g.f.}}(K, \beta, p) \tag{2.2.4}$$

Here

$$|Z^{p-1}(\mathbf{K})| = \frac{|C^{p-1}(K)|}{|B^p(K)|} \tag{2.2.5}$$

since $C^{p-1}(K)$ is a direct sum of the group $Z^{p-1}(K)$ and of a group isomorphic to $B^p(K)$. We define also the partition functions

$$\Xi^{\mathrm{g.f.}}(K, \beta, p) = \sum_{z \in Z^p(K)} e^{-\beta H^p_K(z)}$$
$$\Xi(K, \beta, p) = |Z^{p-1}(K)|\ \Xi^{\mathrm{g.f.}}(K, \beta, p)$$

Hereafter, we table the convention that, as in (2.2.1), the boundary operator is restricted to the considered complex.

To introduce the duality transformation, we consider the Fourier expansion of $e^{\beta\delta(\alpha)}$ (identifying the dual group of $\mathbb{Z}_q$ with $\mathbb{Z}_q$):

$$\begin{aligned} e^{\beta\delta(\alpha)} = 1 + (e^\beta - 1)\,\delta(\alpha) &= \sum_{n=0}^{q-1} \left[\delta(n) + \frac{e^\beta - 1}{q}\right] e^{(2i\pi/q)n\alpha} \\ &= \frac{e^\beta - 1}{q}\left(1 + \frac{q}{e^\beta - 1}\right)^{\delta(n)} e^{(2i\pi/q)n\alpha} \end{aligned}$$

Applying this formula to every p-cell in (2.2.1), denoting by $N^p(K)$ the number of p-cells in K, and introducing the scalar product $(\varphi, \psi) = \sum_{s^p \in K}^{(+)} \varphi(s^p)\,\psi(s^p)$ for every two $\mathbb{Z}_q$-valued p-chains, we get

$$\mathbf{Z}(K, \beta, p) = \left(\frac{e^\beta - 1}{q}\right)^{N^p(K)} \sum_{\varphi \in C^p(K)} e^{-\beta^* H_K^p(\varphi)} \sum_{\sigma \in C^{p-1}(K)} e^{(2i\pi/q)(\varphi, d\sigma)}$$

Here the dual temperature β^* is defined by

$$(e^{\beta^*} - 1)(e^\beta - 1) = q \tag{2.2.6}$$

Observing that

$$(\varphi, d\sigma) = (d^*\varphi, \sigma) \quad \left[= \sum_{S^{p-1} \in K}^{(+)} \varphi(\partial^* s^{p-1})\,\sigma(s^{p-1}) \right]$$

where the boundary and coboundary operator are restricted to K, and summing over σ^{p-1}, we get

$$\mathbf{Z}(K, \beta, p) = \left(\frac{e^\beta - 1}{q}\right)^{N^p(K)} q^{N^{p-1}(K)} \sum_{z \in Z_p(K)} e^{-\beta^* H_K^p(z)}$$

Using the isomorphism between $Z_p(K)$ and $Z^{d-p}(K^*)$, where K^* is the dual complex of K, and the correspondence (A.4), we obtain

$$\mathbf{Z}(K, \beta, p) = \left(\frac{e^\beta - 1}{q}\right)^{N^p(K)} q^{N^{p-1}(K)} \sum_{z_* \in Z^{d-p}(K^*)} \exp[-\beta^* H_{K^*}^{d-p}(z_*)]$$

with $H_{K^*}^{d-p}$ defined in the same way as in (2.2.2). Thus,

$$\mathbf{Z}(K, \beta, p) = \left(\frac{e^\beta - 1}{q}\right)^{N^p(K)} q^{N^{p-1}(K)}\, \Xi^{\text{g.f.}}(K^*, \beta^*, d-p) \tag{2.2.7}$$

Introducing the factor

$$\omega(K^*, \beta^*, d-p) = \frac{\Xi^{\text{g.f.}}(K^*, \beta^*, d-p)}{\mathbf{Z}^{\text{g.f.}}(K^*, \beta^*, d-p)}$$

taking into account (2.2.4) and (2.2.5), we finally get the duality relation

$$\mathbf{Z}(K, \beta, p) = \left(\frac{e^\beta - 1}{q}\right)^{N^p(K)} q^{N^{p-1}(K) - N^{p+1}(K)} |B_p(K)|\, \omega(K^*, \beta^*, d-p) \times \mathbf{Z}(K^*, \beta^*, d-p) \tag{2.2.8}$$

Note that if the cell complex K has a trivial p-homology group, $H_p(K) = H^{d-p}(K^*) = \{0\}$, the factor $\omega(K^*, \beta^*, d-p)$ in (2.2.8) is equal to one.

We introduce the notation

$$[\cdot](K, \beta, p) = \sum_{\sigma \in C^{p-1}(K)} e^{-\beta H_K^p(d\sigma)}$$

in terms of which an expectation value equals

$$\langle \cdot \rangle(K, \beta, p) = [\mathbf{Z}(K, \beta, p)]^{-1} [\cdot](K, \beta, p)$$

Submitting the sum in $[\cdot](K, \beta, p)$ to a similar procedure as in the proof of (2.2.8), we get, supposing that $H_p(K) = \{0\}$, the following identities, which relate, in particular, the free "f" and "ord" b.c.:

$$\left\langle \prod_{s^p \in P} \delta[d\sigma(s^p)] \right\rangle (K, \beta, p)$$
$$= (1 - e^{-\beta})^{-N^p(P)} \left\langle \prod_{s^{d-p} \in P^*} e^{-\beta^* \delta[d\sigma_*(s^{d-p})]} \right\rangle (K^*, \beta^*, d-p) \qquad (2.2.9)$$

where P is a subcomplex of K.

We end this subsection with a technical statement that will be crucial in obtaining noninteracting contour models.

Lemma 2.2.1. Let L and I be two subcomplexes of $\mathbb{L}$ such that the $\mathbb{Z}_q$-valued p-cohomology group of K is trivial, $H^p(K) = \{0\}$, I is open, and $I \subset K$; then, for $1 \leqslant p \leqslant d-1$

(a) $$[e^{\beta N^p(K \setminus I)} |Z^{p-1}(K)|]^{-1} \left\{ \prod_{s^p \in K \setminus I} \delta[d\sigma(s^p)] \right\} (K, \beta, p) = \Xi^{\text{g.f.}}(I, \beta, p)$$

(b) Let the f_I be a function on C^p with support in $I_0 \subset I$; then

$$[e^{\beta N^p(K \setminus I)} |Z^{p-1}(K)|]^{-1} \left\{ \prod_{s^p \in K \setminus I} \delta[d\sigma(s^p)]\, f_I(d\sigma) \right\} (K, \beta, p)$$
$$= \sum_{z \in Z^p(I)} f_I(z)\, e^{-\beta H_I^p(z)}$$

Proof. Consider the canonical extension i_I: $C^p(I) \to C^p(K)$. Since I is open in K, $d_K i_I = i_I d_I$ (d_X denoting the restriction of d to the complex X) and we get

$$i_I Z^p(I) = \{z \in Z^p(K) \mid z(s^p) = 0 \text{ if } s^p \in K \setminus I\}$$

Since $H^p(K)=\{0\}$ by assumption we have

$$i_I Z^p(I)=\{b\in B^p(K)\,|\,b(s^p)=0 \text{ if } s^p\in K\backslash I\}$$

Hence

$$\begin{aligned}\Xi^{\text{g.f.}}(I,\beta,p)&=e^{-\beta N^p(K\backslash I)}\sum_{b\in B^p(K)} e^{-\beta H_K^p(b)}\prod_{s^p\in K\backslash I}\delta[b(s^p)]\\&=[e^{\beta N^p(K\backslash I)}\,|Z^{p-1}(K)|]^{-1}\left\{\prod_{s^p\in K\backslash I}\delta[d\sigma(s^p)]\right\}(K,\beta,p)\end{aligned}$$

The proof of statement (b) is analogous. ∎

2.3. Definition of Contours

To introduce contours, we shall use the concepts of the *envelope* and of the *fringe* of a set of lattice p-cells Q^p, $Q^p\subset\mathbb{L}^p$ ($\mathbb{L}^p$ denotes the set of p-cells in $\mathbb{L}$). We shall denote by $\bar{Q}^p$ the closure of Q^p and we define:

- The *envelope* $E(Q^p)$ of Q^p as the maximal closed subcomplex of $\mathbb{L}$ whose set of r-cells, $r\leqslant p$, coincides with $\bar{Q}^p$, $E(Q^p)\cap\mathbb{L}^p=Q^p$. An explicit expression is $E(Q^{p})=\bigcup_{q=p+1}^{d}E^q(Q^p)\cup\bar{Q}^p$, with $E^p(Q^p)=Q^p$ and $E^q(Q^p)=\{s^q\in\mathbb{L}^q\,|\,\text{all } s^{q-1} \text{ of } \partial s^q \text{ belongs to } E^{q-1}(Q^p)\}$ whenever $q\geqslant p+1$.
- The *fringe* $F(Q^p)$ of Q^p by $F(Q^p)=\mathbb{L}\backslash\{E(Q^p)\cup E(\mathbb{L}^0\backslash\mathbb{L}^0\cap E(Q^p)\}$.
- The *boundary* $B(K)$ of a cell complex K by $B(K)=\overline{\mathbb{L}\backslash K}\cap K$.

In particular, if $A\subset\mathbb{L}^0$, the complement of union of envelopes of A and $A^c=\mathbb{L}^0\backslash A$ is then called the fringe $F(A)$ of A, $F(A)=L\backslash[E(A)\cup E(A^c)]$. Notice that the complex $F(A)$ contains no lattice points. $F(A)\cap\mathbb{L}^0=\varnothing$, and that $F(A)=F(A^c)$. The contours will have for their support the fringe of their exterior.

Consider thus a configuration $\sigma\in C^{p-1}(\mathbb{L})$ such that the set of p-cells $M^p(\sigma)=\{s^p\in\mathbb{L}^p\,|\,\sigma(\partial s^p)\neq 0\}$ is finite. Denoting by $Q^p(\sigma)$ the unique infinite component (connected subcomplex) of the complex $\mathbb{L}^p\backslash M^p(\sigma)$, we shall denote by $F(\sigma)$ the fringe of $Q^p(\sigma)$: $F(\sigma)=F(Q^p(\sigma))$. Clearly, $s^p\in Q^p(\sigma)$ implies $\sigma(\partial s^p)=0 \pmod q$, while $s^p\in F(\sigma)$ implies $\sigma(\partial s^p)\neq 0$ (mod q).

A pair $\tilde{\gamma}=\{\gamma,\sigma_\gamma\}$, where γ is a component of $F(\sigma)$ and σ_γ the restriction of σ on the complex γ, will be called *an external contour of* σ. A pair $\tilde{\gamma}=\{\gamma,\sigma_\gamma\}$ with γ a subcomplex of $\mathbb{L}$ and σ_γ a configuration on it, $\sigma\in C^{p-1}(\gamma)$, will be called a $(p-1)$-contour if there exists a configuration $\sigma\in C^{p-1}(\gamma)$ such that γ is its external contour.

Whenever $\tilde{\gamma}$ is a contour, we call the complex γ its *support*, $\gamma=\operatorname{supp}\tilde{\gamma}$, and introduce the complexes $\mathrm{Ext}\,\gamma$ as the unique infinite component of $\mathbb{L}\backslash\gamma$, $V(\gamma)=\mathbb{L}\backslash\mathrm{Ext}\,\gamma$, and $\mathrm{Int}\,\gamma=V(\gamma)\backslash\gamma$. The complexes γ and $V(\gamma)$ are open, whereas the complexes $\mathrm{Int}\,\gamma$ and $\mathrm{Ext}\,\gamma$ are closed. Notice that in the case $p=1$ (scalar model) there are no lattice sites in γ; then a contour may be identified with its support without any configuration σ_γ on it.

Two contours $\tilde{\gamma}_1$ and $\tilde{\gamma}_2$ with disjoint supports are called *mutually compatible*. They are called *mutually compatible external contours* if $V(\gamma_1)\subset\mathrm{Ext}\,\gamma_2$ and $V(\gamma_2)\subset\mathrm{Ext}\,\gamma_1$. It is easy to show that whenever $\tilde{\theta}=\{\tilde{\gamma}_1,\tilde{\gamma}_2,\tilde{\gamma}_3,\dots,\tilde{\gamma}_n\}$ is a family of mutually external contours, there exists a configuration σ with the same set of external contours. For such a family $\tilde{\theta}$ of external contours we shall use the notation $\theta=\mathrm{supp}\,\tilde{\theta}=\bigcup_i\gamma_i$, $V(\theta)=\bigcup_i V(\gamma_i)$, $\mathrm{Int}\,\theta=V(\theta)\backslash\theta$, $\mathrm{Ext}\,\theta=\mathbb{L}\backslash V(\theta)$, and $\mathrm{Ext}_V\theta=V\cap\mathrm{Ext}\,\theta$ whenever $V\subset\mathbb{L}$.

For contours on the dual lattice $\mathbb{L}^*$ defined in the same way as above with $\mathbb{L}$ replaced by $\mathbb{L}^*$ we shall use the notation $\tilde{\gamma}_*=\{\gamma_*,\sigma_*\gamma_*\}$. Notice that γ_* is a support of a contour and hence an *open subcomplex* of $\mathbb{L}^*$, while γ^* is the dual of the complex γ and it is thus a *closed subcomplex* of $\mathbb{L}^*$ for every $\tilde{\gamma}=\{\gamma,\sigma_\gamma\}$.

2.4. Inductive Expression of Partition Functions

In accordance with our strategy outlined in Section 2.1, we first express partition functions in terms of contours on both $\mathbb{L}$ and its dual $\mathbb{L}^*$. We proceed in an inductive way. When doing so, we will meet partition functions with "disordered" and "ordered" boundary conditions. The partition functions with "disordered" boundary conditions are defined for a subcomplex $V(\theta)$ whenever θ is a support of a family of mutually external contours by

$$\Xi^{\mathrm{g.f.}}(\theta,\beta,p\,|\,\mathrm{dis})=\sum_{z\in Z^p(V(\theta))}e^{-\beta H^p_{V(\theta)}(z)}\prod_{s^p\in\theta}[1-\delta(z(s^p))]$$

$$\Xi(\theta,\beta,p\,|\,\mathrm{dis})=|Z^{p-1}(V(\theta))|\,\Xi^{\mathrm{g.f.}}(\theta,\beta,p\,|\,\mathrm{dis})$$

Notice that one has

$$\Xi^{\mathrm{g.f.}}(\theta,\beta,p\,|\,\mathrm{dis})=\prod_{\gamma\in\theta}\Xi^{\mathrm{g.f.}}(\gamma,\beta,p\,|\,\mathrm{dis})$$

The partition functions $\Xi^{\mathrm{g.f.}}(\theta_*,\beta^*,d-p\,|\,\mathrm{dis})$ and $\Xi(\theta_*,\beta^*,d-p\,|\,\mathrm{dis})$ for a subcomplex $V(\theta_*)$ of $\mathbb{L}^*$ are defined in an analogous way.

Remark. In the case $p=1$ we have

$$\Xi^{\mathrm{g.f.}}(\theta,\beta,p\,|\,\mathrm{dis})=\sum_{\sigma\in C^0(V(\theta))}e^{-\beta H^1_{V(\theta)}(d\sigma)}\prod_{s^1\in\theta}[1-\delta(d\sigma(s^1))]\quad(2.4.1)$$

where d is restricted to the complex $V(\theta)$. The proof of this fact follows from the Alexander theorem.

Indeed, by construction of contours, $\mathbb{L}\backslash V(\theta)$ has only one connected component $[\pi^0(\mathbb{L}\backslash V(\theta))=1]$; then, from Theorem A.1 (Appendix A), $\pi^{d-1}([V(\theta)]^*)$ equals zero and we obtain

$$\pi^1(V(\theta))=\pi^{d-1}([V(\theta)]^*)=0$$

Moreover,

$$\tau^0(V(\theta))=\tau^{d-1}([V(\theta)]^*)=0, \qquad \tau^1(V(\theta))=\tau^{d-2}([V(\theta)]^*)=0$$

since the closed complex $[V(\theta)]^*$ is $d-1$ and $d-2$ torsion free. Therefore we deduce from (A.7) that $H^1(V(\theta))$ is the null group.

On the other hand, for open complexes $V=V(\theta)$

$$\frac{|C^0(V)|}{|B^1(V)|}=|Z^0(V)|=1$$

This is a consequence of the fact that $B^0(V)$ is the null group for open complexes (ref. 26, Vol. 2, p. 94) and that $H^0(V)$ is isomorphic to $H^d(V^*)$, which is also the null group, since $\pi^d(V^*)=0$ and the closed complex V^* is $d-1$ and $d-2$ torsion free according to Theorem A.1.

The equality (2.4.1) then follows from (2.2.4).

The following statement yields the sought inductive expression and serves thus as a starting point for expanding partition functions in terms of contour models. This statement plays in our case the role of Lemma 2.3 from ref. 27.

Lemma 2.4.1. Whenever V and V_* are open subcomplexes of $\mathbb{L}$ and $\mathbb{L}^*$, respectively, γ is the support of a $(p-1)$-contour of $\mathbb{L}$, and γ_* is the support of a $(d-p-1)$-contour of $\mathbb{L}^*$, we have

$$\Xi^{\text{g.f.}}(V,\beta,p)=\sum_{\theta\subset V}\{\exp[\beta N^p(\text{Ext}_V\theta)]\}\,\Xi^{\text{g.f.}}(\theta,\beta,p\,|\,\text{dis})$$

$$\Xi^{\text{g.f.}}(\gamma,\beta,p\,|\,\text{dis})=g(\gamma,\beta,p)\left(\frac{e^{\beta}-1}{q}\right)^{N^p(\text{Int}\,\gamma)}|Z^p(V(\gamma))|$$
$$\times\,\Xi^{\text{g.f.}}([\text{Int}\,\gamma]^*,\beta^*,d-p)$$

$$\Xi^{\text{g.f.}}(V_*,\beta^*,d-p)=\sum_{\theta_*\subset V_*}\{\exp[\beta^*N^{d-p}(\text{Ext}_{V^*}\theta_*)]\}$$
$$\times\,\Xi^{\text{g.f.}}(\theta_*,\beta^*,d-p\,|\,\text{dis})$$

$$\Xi^{\text{g.f.}}(\gamma_*,\beta^*,d-p\,|\,\text{dis})=g(\gamma_*,\beta^*,d-p)\left(\frac{e^{\beta^*}-1}{q}\right)^{N^{d-p}(\text{Int}\,\gamma_*)}$$
$$\times\,|Z^{d-p}(V(\gamma_*))|\,\Xi^{\text{g.f.}}([\text{Int}\,\gamma_*]^*,\beta,p)$$

with

$$g(\gamma, \beta, p)=\frac{\Xi(\gamma, \beta, p \mid \text{dis})}{|H^{p}(V(\gamma))|\, q^{N^{p-1}}(\gamma)\, \mathbf{Z}(\operatorname{Int}\gamma, \beta, p)}\leqslant 1$$

$$g(\gamma_{*}, \beta^{*}, d-p)=\frac{\Xi(\gamma_{*}, \beta^{*}, d-p \mid \text{dis})}{|H^{d-p}(V(\gamma_{*}))|\, q^{N^{d-p-1}}(\gamma_{*})\, \mathbf{Z}(\operatorname{Int}\gamma_{*}, \beta^{*}, d-p)}\leqslant 1$$

The sums are over all supports of families of mutually external contours such that $V(\theta)\cap \mathbb{L}^{p-1}\subset V$, $V(\theta_{*})\cap (\mathbb{L}^{*})^{d-p-1}\subset V_{*}$.

Proof. We first consider the first statement and let K denote a subcomplex of $\mathbb{L}$ satisfying the conditions in Lemma 2.1.1 with $I=V$. From the statement (a) of Lemma 2.2.1 it follows that

$$\Xi^{\text{g.f.}}(V, \beta, p)=[e^{\beta N^{p}(K\setminus V)}\, |Z^{p-1}(K)|]^{-1}\left\{\prod_{s^{p}\in K\setminus V}\delta[d\sigma(s^{p})]\right\}(K, \beta, p)$$

Let

$$\chi^{p}_{\theta;K}=\prod_{s^{p}\in K\setminus V(\theta)}\delta[d\sigma(s^{p})]\prod_{s^{p}\in\theta}\{1-\delta[d\sigma(s^{p})]\}$$

where d is restricted to K; then it is clear that we can do the following expansion:

$$\left\{\prod_{s^{p}\in K\setminus V}\delta[d\sigma(s^{p})]\right\}(K, \beta, p)=\sum_{\theta\subset V}[\chi^{p}_{\theta;K}](K, \beta, p)$$

From the statement (b) of Lemma 2.2.1 we infer

$$[e^{\beta N^{p}(K\setminus V(\theta))}\, |Z^{p-1}(K)|]^{-1}[\chi^{p}_{\theta;K}](K, \beta, p)=\Xi^{\text{g.f.}}(\theta, \beta, p \mid \text{dis})$$

which gives the result.

In the case $p=1$ and if we assume that the cohomology group $H^{1}(K)$ is trivial, we can prove this result in the following way: in this case,

$$\Xi^{\text{g.f.}}(V, \beta, 1)=\mathbf{Z}^{\text{g.f.}}(V, \beta, 1)=\mathbf{Z}(V, \beta, 1) \tag{2.4.2}$$

The last equality follows from the fact for open complex V, $|Z^{0}(V)|=1$; then it is clear that

$$\begin{aligned}\mathrm{Z}(V, \beta, 1)&=\sum_{\sigma\in C^{0}(K)}\sum_{\theta\subset V}e^{-\beta H^{1}_{V}(d\sigma)}\prod_{s^{0}\in V\setminus V(\theta)}\delta[\sigma(s^{0})]\prod_{s^{1}\in\theta}\{1-\delta[d\sigma(s^{1})]\}\\&=\sum_{\theta\subset V}e^{\beta N^{1}(\mathrm{Ext}_{V}\theta)}\sum_{\sigma\in C^{0}(V(\theta))}e^{-\beta H^{1}_{V(\theta)}(d\sigma)}\prod_{s^{1}\in\theta}\{1-\delta[d\sigma(s^{1})]\}\end{aligned}$$

we then use (2.4.1).

We now consider the second statement; we have by definition

$$\begin{aligned}
&\Xi^{\text{g.f.}}(\gamma, \beta, p \mid \text{dis}) \\
&\quad = [|Z^{p-1}(V(\gamma))|]^{-1}\, \Xi(\gamma, \beta, p \mid \text{dis}) \\
&\quad = [|Z^{p-1}(V(\gamma))|]^{-1}\, g(\gamma, \beta, p)\, |H^p(V(\gamma))|\, q^{N^{p-1}(\gamma)}\, \mathbf{Z}(\text{Int}\,\gamma, \beta, p))
\end{aligned}$$

The proof then follows from relations (2.2.7) and (2.2.5).

The third and fourth statements are proved analogously.

The proof of the bounds on $g(\gamma, \beta, p)$ and $g(\gamma_*, \beta^*, d-p)$ relies on the fact that

$$\sum_{\sigma \in C^{p-1}(\text{Int}\,\gamma)} e^{-\beta H^p_{\text{Int}\,\gamma}(d\sigma + h)} \leqslant \sum_{\sigma \in C^{p-1}(\text{Int}\,\gamma)} e^{-\beta H^p_{\text{Int}\,\gamma}(d\sigma)}$$

for any $h \in C^p(\text{Int}\,\gamma)$, which one may easily show by a duality transformation. ∎

In fact, to have an even closer analog of Lemma 2.3 from ref. 27, we introduce diluted and crystal partition functions by

$$\begin{aligned}
\mathbf{Z}^{\text{dil}}(V, \beta, p) &= \Xi^{\text{g.f.}}(\mathring{V}, \beta, p) \\
\mathbf{Z}^{\text{cryst}}(\gamma, \beta, p) &= \Xi^{\text{g.f.}}(\gamma, \beta, p \mid \text{dis}) \\
\mathbf{Z}^{\text{cryst}}(\theta, \beta, p) &= \prod_{\gamma \in \theta} \mathbf{Z}^{\text{cryst}}(\gamma, \beta, p) \\
\mathbf{Z}^{\text{dil}}(V_*, \beta^*, d-p) &= [(e^{\beta} - 1)\, q^{p/d-1}]^{N^{d-p}(\mathring{V}_*)}\, \mathbf{Z}^{\text{g.f.}}(\mathring{V}_*, \beta^*, d-p) \\
\mathbf{Z}^{\text{cryst}}(\gamma_*, \beta^*, d-p) &= [(e^{\beta} - 1)\, q^{p/d-1}]^{N^{d-p}(V(\gamma_*))}\, \Xi^{\text{g.f.}}(\gamma_*, \beta^*, d-p \mid \text{dis}) \\
\mathbf{Z}^{\text{cryst}}(\theta_*, \beta^*, d-p) &= \prod_{\gamma_* \in \theta_*} \mathbf{Z}^{\text{cryst}}(\gamma_*, \beta^*, d-p)
\end{aligned} \tag{2.4.3}$$

where V and V_* are (open or closed) subcomplexes of $\mathbb{L}$ and $\mathbb{L}^*$, respectively. The open subcomplexes $\mathring{V}$ and $\mathring{V}_*$ are defined as $\mathring{V} = V \backslash B(V)$ and $\mathring{V}_* = V_* \backslash B(V_*)$ ($\mathring{V} = V$ if V is open). Hence both $\mathbf{Z}^{\text{dil}}(V, \beta, p)$ and $\mathbf{Z}^{\text{dil}}(V_*, \beta^*, d-p)$ are partition functions with ordered boundary condition.

Under the above definitions Lemma 2.4.1 reads as follows.

Lemma 2.4.2.

$$\mathbf{Z}^{\text{dil}}(V, \beta, p) = \sum_{\theta \subset V} \{\exp[\mu N^p(\mathring{V} \backslash V(\theta))]\}\, \mathbf{Z}^{\text{cryst}}(\theta, \beta, p)$$

$$\mathbf{Z}^{\text{cryst}}(\gamma, \beta, p) = g(\gamma, \beta, p)\, D(\gamma, p)\, \mathbf{Z}^{\text{dil}}([\text{Int}\,\gamma]^*, \beta^*, d-p)$$

$$\mathbf{Z}^{\text{dil}}(V_*, \beta^*, d-p) = \sum_{\theta_* \subset V_*} \{\exp[\mu_* N^{d-p}(\mathring{V}_* \setminus V(\theta_*))]\} \times \mathbf{Z}^{\text{cryst}}(\theta_*, \beta^*, d-p)$$

$$\mathbf{Z}^{\text{cryst}}(\gamma_*, \beta^*, d-p) = g(\gamma_*, \beta^*, d-p)\, D(\gamma_*, d-p)\, \mathbf{Z}^{\text{dil}}([\text{Int}\,\gamma_*]^*, \beta, p)$$

where

$$D(\gamma, p) = q^{-(p/d) N^p(\text{Int}\,\gamma)}\, |Z^p(V(\gamma))|$$

$$D(\gamma_*, d-p) = (e^\beta - 1)^{N^{d-p}(\gamma_*)}\, q^{-(1-p/d)\, N^{d-p}(V(\gamma_*))}\, |Z^{d-p}(V(\gamma_*))|$$

$$\mu = \beta, \qquad \mu_* = \beta^* + \log[(e^\beta - 1) q^{p/d-1}]$$

The sums are over all supports of families of mutually external contours such that $V(\theta) \cap \mathbb{L}^{p-1} \subset \mathring{V}$. $V(\theta_*) \cap (\mathbb{L}^*)^{d-p-1} \subset \mathring{V}_*$.

The prefactor in the definition of $\mathbf{Z}^{\text{dil}}(V_*, \beta^*, d-p)$ assures that both diluted partition functions yield the free energy $f_p(\beta H)$ of the (gauge fixed) model with Hamiltonian H^p at the temperature β:

$$\begin{aligned} f_p(\beta H) &= \lim_{V \uparrow \mathbb{L}} \frac{1}{N^p(V)} \log \mathbf{Z}^{\text{dil}}(V, \beta, p) \\ &= \lim_{V_* \uparrow \mathbb{L}^*} \frac{1}{N^{d-p}(V_*)} \log \mathbf{Z}^{\text{dil}}(V_*, \beta^*, d-p) \end{aligned} \tag{2.4.4}$$

The limits are over complexes approaching $\mathbb{L}$ (resp. $\mathbb{L}^*$) in the van Hove sense:

$$\frac{N^p(B(\bar{V}))}{N^p(\bar{V})} \to 0 \qquad \left(\text{resp.}\ \frac{N^{d-p}(B(\bar{V}_*))}{N^{d-p}(\bar{V}_*)} \to 0\right)$$

The proof of the equality of the two limits in (2.4.4) is based on the duality formula (2.2.7) and is given in Appendix B.

2.5. Partition Functions in Terms of Contour Models

Our aim is to express partition functions $\mathbf{Z}^{\text{dil}}(V, \beta, d-p)$ and $\mathbf{Z}^{\text{dil}}(V_*, \beta^*, d-p)$ (and corresponding probabilities of external contours) in terms of two contour models ϕ and ϕ_* living on supports of $(p-1)$-contours of $\mathbb{L}$ and on supports of $(d-p-1)$-contours on $\mathbb{L}^*$, respectively. The former will describe the low-temperature phases, while the latter, taking into account the duality between ordered and free boundary conditions, will yield some information about the high-temperature disordered phase (of the same model). As in the ordinary PS theory, it is actually copnvenient to introduce contour models with parameters b and b_*, respectively. The transition temperature β_t will then be identified with the unique temperature for which $b = b_* = 0$ and will turn out to be near β_0 fixed by the equation $\mu = \mu_*$.

The partition function, in a volume V, of a contour model ϕ with contour weights $\phi(\gamma)$ is given by

$$\mathscr{Z}(V \mid \phi) = \sum_{\partial \subset V} \prod_{\gamma \in \partial} \phi(\gamma)$$

where the sum is over all compatible families ∂ of supports of contours in V such that $V(\gamma) \cap \mathbb{L}^{p-1} \subset \mathring{V}$. The power of contour models stems from the fact that the compatibility is defined pairwise, $\gamma \cap \gamma' = \varnothing$, for each pair $\gamma, \gamma' \in \partial$, which implies that skipping any contour γ from ∂ leaves the family $\partial \backslash \gamma$ again compatible.

While we refer to, e.g., refs. 8 or 25 or Appendix B of ref. 28 for details of the theory of contour models in a form suitable for our purposes, we just recall here that introducing

$$\mathscr{Z}(\gamma \mid \phi) = \phi(\gamma)\, \mathscr{Z}(\operatorname{Int} \gamma \mid \phi)$$

one has

$$\mathscr{Z}(V \mid \phi(= \sum_{\theta \subset V} \prod_{\gamma \in \theta} \mathscr{Z}(\gamma \mid \phi) = \sum_{\theta \subset V} \mathscr{Z}(\theta \mid \phi)$$

with the sum over families of mutually external contours. Partition functions of a contour model ϕ on $(p-1)$-contours on $\mathbb{L}$ with a parameter b are defined by

$$\mathscr{Z}(V \mid \phi, b) = \sum_{\theta \subset V} e^{bN^p(V(\theta))}\, \mathscr{Z}(\theta \mid \phi)$$

and those of a contour model ϕ_* on $(d-p-1)$-contours on $\mathbb{L}^*$ with a parameter b_* by

$$\mathscr{Z}(V_* \mid \phi_*, b_*) = \sum_{\theta_* \subset V_*} e^{b_* N^{d-p}(V(\theta_*))}\, \mathscr{Z}(\theta_* \mid \phi_*)$$

To have a good control on partition functions of contour models, one relies on their "τ-functionality"; in our case contour models ϕ and ϕ_* are called τ-functionals if they satisfy the estimates

$$|\phi(\gamma)| \leqslant e^{-\tau N^p(\gamma)}, \qquad |\phi_*(\gamma_*)| \leqslant e^{-\tau N^{d-p}(\gamma_*)} \tag{2.5.1}$$

for every γ and γ_*. Here τ is a fixed constant depending only on d and p. In particular, if ϕ and ϕ_* are τ-functionals, the limits

$$f(\phi)=\lim \frac{1}{N^p(V)} \log \mathscr{Z}(V\mid\phi), \qquad f(\phi_*)=\lim \frac{1}{N^{d-p}(V_*)} \log \mathscr{Z}(V_*\mid\phi_*)$$

exist and the boundary terms

$$\begin{aligned} \sigma(V\mid\phi) &= \log \mathscr{Z}(V\mid\phi) - N^p(V)\, f(\phi) \\ \sigma(V_*\mid\phi_*) &= \log \mathscr{Z}(V_*\mid\phi_*) - N^{d-p}(V_*)\, f(\phi_*) \end{aligned} \tag{2.5.2}$$

may be evaluated by

$$|\sigma(V\mid\phi)| \leqslant N^p(B(\bar{V}))e^{-\tau}, \qquad |\sigma(V_*\mid\phi_*)| \leqslant N^{d-p}(B(\bar{V}_*)e^{-\tau} \tag{2.5.3}$$

Now we may formulate our main statement about the equivalence with contour models which will serve as a basis for proving different theorems from Section 1. Its assumption may be considered as an extension of the Peierls condition from Pirogov–Sinai theory to our case.

Proposition 2.5.1. Let $1 \leqslant p \leqslant d-1$ and suppose that (q is such that)

$$q^{-(p/d)N^p(V(\gamma))}\, |Z^p(V(\gamma))| \leqslant e^{-2N^p(\gamma)} \tag{2.5.4}$$

and

$$q^{[-(d-p)/d]N^{d-p}(V(\gamma_*))}\, |Z^{d-p}(V(\gamma_*))| \leqslant e^{-2\tau N^{d-p}(\gamma_*)} \tag{2.5.5}$$

Then for every β there exist τ-functionals ϕ and ϕ_* and parameters b and b_* such that

$$b+\mu+f(\phi)=b_*+\mu_*+f(\phi_*)=f_p(\beta H) \tag{2.5.6}$$

and

$$\mathscr{Z}(\gamma\mid\phi)=e^{-(b+\mu)\,N^p(V(\gamma))}\,\mathbf{Z}^{\text{cryst}}(\gamma,\beta,p) \tag{2.5.7}$$

for every $(p-1)$-contour γ on $\mathbb{L}$, and

$$\mathscr{Z}(\gamma_*\mid\phi_*)=\{\exp[-b_*+\mu_*)\,N^{d-p}(V(\gamma_*))]\}\,\mathbf{Z}^{\text{cryst}}(\gamma_*,\beta^*,d-p) \tag{2.5.8}$$

for every $(d-p-1)$-contour γ_* on $\mathbb{L}^*$.

Moreover, $\min(b, b_*)=0$ and there exists a unique temperature $\beta_t(q, p, d)$ such that for $\beta \in [0, \beta_t)$ one has $b>0$ and $b_*=0$, $b=b_*=0$ for $\beta=\beta_t$ and $b=0$, $b_*>0$ for $\beta \in (\beta_t, \infty)$.

Remarks.

1. Verifying the hypothesis of the proposition (generalized Peierls condition) involves an estimate on the cardinality of the group $Z^p(V(\gamma))$, resp. $Z^{d-p}(V(\gamma_*))$. This may be a complicated topological question and we shall verify (2.5.4) and (2.5.5) (for q large enough) only in particular cases, namely, for $p=1$ (scalar model) with $d \geqslant 2$ and $p=2$ (gauge model) for $d=3$ (cf. Appendix B). For $p=2$ with $d=4$, these hypotheses have been very recently proved in ref. 14.

2. The equalities (2.5.7) and (2.5.8) lead immediately to

$$\begin{aligned}\mathscr{Z}(V \mid \phi, b) &= \{\exp[-\mu N^p(\mathring{V})]\}\, \mathbf{Z}^{\mathrm{dil}}(V, \beta, p)\\ &= (\{\exp[-\beta N^p(\mathring{V})]\}\, \Xi^{\mathrm{g.f.}}(\mathring{V}, \beta, p))\end{aligned} \tag{2.5.9}$$

and

$$\begin{aligned}\mathscr{Z}(V_* \mid \phi_*, b_*) &= \{\exp[-\mu_* N^{d-p}(\mathring{V}_*)]\}\, \mathbf{Z}^{\mathrm{dil}}(V_*, \beta^*, d-p)\\ &= (\{\exp[-\beta^* N^{d-p}(\mathring{V}_*)]\}\, \Xi^{\mathrm{g.f.}}(\mathring{V}_*, \beta^*, d-p))\end{aligned} \tag{2.5.10}$$

This combined with (2.5.7) and (2.5.8) imply that the contour model ϕ with parameter b (resp. ϕ_* with b_*) reproduces the probabilities of external contours governed by the Hamiltonian H^p (resp. its dual) under ordered boundary conditions. This will eventually allow us to distinguish ordered and disordered phases by evaluating the expectation $\langle \delta[\sigma^p(s^p)] \rangle$ for some fixed s^p.

Proof. We shall construct ϕ and ϕ_* following the inductive procedure from ref. 25. First of all one observes that for every $\alpha, \alpha_* \geqslant 0$ one may define, by induction in $N^p(V(\gamma))$ and $N^{d-p}(V(\gamma_*))$, contour functionals ϕ^α, $\phi_*^{\alpha_*}$ satisfying

$$\mathscr{Z}(\gamma \mid \phi^\alpha) = \{\exp[-(\alpha+\mu)\, N^p(V(\gamma))]\}\, \mathbf{Z}^{\mathrm{cryst}}(\gamma, \beta, p) \tag{2.5.11}$$

$$\mathscr{Z}(\gamma_* \mid \phi_*{}^{\alpha_*}) = \{\exp[-(\alpha_*+\mu_*)\, N^{d-p}(V(\gamma_*))]\, \mathbf{Z}^{\mathrm{cryst}}(\gamma_*, \beta^*, d-p) \tag{2.5.12}$$

Let us turn these functionals in an artificial way into τ-functionals by defining

$$\bar{\phi}^\alpha(\gamma) = \min(\phi^\alpha(\gamma), e^{-\tau N^p(\gamma)})$$

$$\bar{\phi}_*^{\alpha_*}(\gamma_*) = \min(\phi_*^{\alpha_*}(\gamma_*), e^{-\tau N^{d-\Delta p}(\gamma_*)})$$

Define further

$$b_* = \sup\{\alpha \mid \alpha + \mu + f(\bar{\phi}^\alpha) \leqslant f_p(\beta H)\}$$
$$b^* = \sup\{\alpha_* \mid \alpha_* + \mu_* + f(\bar{\phi}_*^{\alpha_*}) \leqslant f_p(\beta H)\}$$

We shall show that

$$b + \mu + f(\bar{\phi}^b) = b_* + \mu_* + f(\bar{\phi}_*^{b_*})) = f_p(\beta H) \tag{2.5.13}$$

by proving that $f(\bar{\phi}^\alpha)$ and $f(\bar{\phi}_*^{\alpha_*})$ are continuous in α and α_*. This follows for, say, $f(\bar{\phi}^\alpha)$ from the fact that

$$\frac{1}{N^p(V)} \log \mathscr{Z}(V \mid \bar{\phi}^\alpha)$$

is Lipschitz uniformly in V. Indeed[29]

$$\begin{aligned}
&\left| \frac{1}{N^p(V)} \frac{d}{d\alpha} \log \mathscr{Z}(V \mid \bar{\phi}^\alpha) \right| \\
&\qquad \leqslant \sum_{\gamma : s^p \in \gamma} \left| \frac{d}{d\alpha} \bar{\phi}^\alpha(\gamma) \right| \cdot \left| \frac{\mathscr{Z}(V \backslash [\gamma] \mid \bar{\phi}^\alpha)}{\mathscr{Z}(V \mid \bar{\phi}^\alpha)} \right| \\
&\qquad \leqslant \sum_{\gamma : d^p \in \gamma} e^{-\tau N^p(\gamma)} 2N^p(V(\gamma)) \leqslant ce^{-\tau}
\end{aligned} \tag{2.5.14}$$

Here the sum is over all contours containing a fixed cell s^p and

$$\mathscr{Z}(V \backslash [\gamma] \mid \bar{\phi}^\alpha) = \sum_{\partial \subset V}^* \prod_{\hat{\gamma} \in \partial} \bar{\phi}^\alpha(\hat{\gamma})$$

where the sum $\sum^*$ is over ∂ all compatible with γ.

We use the fact that either $(d/d\alpha)\, \bar{\phi}^\alpha(\gamma) = 0$ or

$$\begin{aligned}
\left| \frac{d}{d\alpha} \bar{\phi}^\alpha(\gamma) \right| &= \left| \frac{d}{d\alpha} \phi^\alpha(\gamma) \right| \\
&= \left| \frac{(d/d\alpha)\, \mathscr{Z}(\gamma \mid \phi^\alpha)\, \mathscr{Z}(\operatorname{Int} \gamma \mid \phi^\alpha) - \mathscr{Z}(\gamma \mid \phi^\alpha)\, (d/d\alpha)\, \mathscr{Z}(\operatorname{Int} \gamma \mid \phi^\alpha)}{[\mathscr{Z}(\operatorname{Int} \gamma \mid \phi^\alpha)]^2} \right| \\
&\leqslant |\bar{\phi}^\alpha(\gamma)| \left\{ \left| \frac{d}{d\alpha} \log \mathscr{Z}(\gamma \mid \phi^\alpha) \right| + \left| \frac{d}{d\alpha} \log \mathscr{Z}(\operatorname{Int} \gamma \mid \phi^\alpha) \right| \right\} \\
&\leqslant 2N^p(V(\gamma))\, |\phi^\alpha(\gamma)|
\end{aligned}$$

since $(d/d\alpha) \log \mathscr{Z}(\gamma \mid \phi^\alpha) = -N^p(V(\gamma))$ according to (2.5.11) and

$$\begin{aligned}
&\left| \frac{d}{d\alpha} \log \mathscr{Z}(\operatorname{Int} \gamma \mid \phi^\alpha) \right| \\
&\quad = \left| \frac{1}{\mathscr{Z}(\operatorname{Int} \gamma \mid \phi^\alpha)} \sum_{\theta \subset \operatorname{Int} \gamma} \frac{d}{d\alpha} \prod_{\hat{\gamma} \in \theta} \mathscr{Z}(\hat{\gamma} \mid \phi^\alpha) \right| \\
&\quad = \left| \frac{1}{\mathscr{Z}(\operatorname{Int} \gamma \mid \phi^\alpha)} \sum_{\theta \subset \operatorname{Int} \gamma} [-N^p(V(\theta))] \prod_{\hat{\gamma} \in \theta} \mathscr{Z}(\hat{\gamma} \mid \phi^\alpha) \right| \leqslant N^p(\operatorname{Int} \gamma)
\end{aligned}$$

using again (2.5.1). Here θ is a family of external contours in $\operatorname{Int} \gamma$.

Now we shall show by induction in $N^p(V(\gamma))$ and $N^{d-p}(V(\gamma_*))$ that ϕ^b and $\phi_*^{b_*}$ are actually τ-functionals, i.e., $\bar{\phi}^b = \phi^b$ and $\bar{\phi}_*^{b_*} = \phi_*^{b_*}$. Suppose that this is known for all $\hat{\gamma}$ with $N^p(V(\hat{\gamma})) \leqslant k$ and all $\hat{\gamma}_*$ with $N^{d-p}(V(\hat{\gamma}_*)) \leqslant k$ and consider a contour γ with $N^p(V(\gamma)) \leqslant k+1$. Notice first that from the induction hypothesis and (2.5.2) one has

$$\begin{aligned}
\mathscr{Z}(\operatorname{Int} \gamma \mid \phi^b) &= \mathscr{Z}(\operatorname{Int} \gamma \mid \bar{\phi}^b) \\
&= \exp[f(\bar{\phi}^b)\, N^p(\operatorname{Int} \gamma) + \sigma(\operatorname{Int} \gamma \mid \bar{\phi}^b)]
\end{aligned}$$

Observing that for any contour $\hat{\gamma}_*$ contributing to $\mathscr{Z}([\operatorname{Int} \gamma]^* \mid \phi_*^{b_*})$ one has $\hat{\gamma}_* \subset [\operatorname{Int} \gamma]^*$, then, using again the induction hypothesis,

$$\begin{aligned}
\mathscr{Z}([\operatorname{Int} \gamma]^* \mid \phi_*^{b_*}) &= \mathscr{Z}([\operatorname{Int} \gamma]^* \mid \bar{\phi}_*^{b_*}) \\
&= \exp\{f(\bar{\phi}_*^{b_*})\, N^{d-p}([\operatorname{Int} \gamma]^*) + \sigma([\operatorname{Int} \gamma]^* \mid \bar{\phi}_*^{b_*})\}
\end{aligned}$$

Starting from the definition (2.5.11), referring to Lemma 2.4.2, to the above observations combined with an obvious inequality

$$\mathscr{Z}([\operatorname{Int} \gamma]^* \mid \phi_*^{b_*}, b_*) \leqslant \{\exp[b_* N^{d-p}([\operatorname{Int} \gamma]^*)]\}\, \mathscr{Z}([\operatorname{Int} \gamma]^* \mid \phi_*{}^{b_*}) \tag{2.5.15}$$

and the equality $N^{d-p}([\operatorname{Int} \gamma]^*) = N^p(\operatorname{Int} \gamma)$, and to the equality (2.5.13), we get

$$\begin{aligned}
\phi^b(\gamma) &= \{\exp[-(b+\mu)\, N^p(V(\gamma))]\}\, \frac{\mathbf{Z}^{\text{cryst}}(\gamma, \beta, p)}{\mathscr{Z}(\operatorname{Int} \gamma \mid \phi^b)} \\
&= g(\gamma, \beta, p)\, D(\gamma, p) \\
&\quad \times \{\exp[-b+\mu)\, N^p(V(\gamma)) + \mu_* N^{d-p}([\operatorname{Int} \gamma]^*)]\} \\
&\quad \times \frac{\mathscr{Z}([\operatorname{Int} \gamma]^* \mid \phi_*^{b_*}, b_*)}{\mathscr{Z}(\operatorname{Int} \gamma \mid \phi^b)}
\end{aligned}$$

$$\begin{aligned}
&\leqslant g(\gamma, \beta, p)\, D(\gamma, p) \exp\{-[b+\mu+f(\bar{\phi}^b)]\, N^p(\operatorname{Int}\gamma) - (b+\mu)\, N^p(\gamma) \\
&\quad + [b_* + \mu_* + f(\bar{\phi}^b_{*^*})]\, N^p(\operatorname{Int}\gamma)\} \\
&\quad \times \exp\{\sigma(\operatorname{Int}\gamma \mid \bar{\phi}^b) + \sigma([\operatorname{Int}\gamma]^* \mid \bar{\phi}^b_{*^*})\} \\
&= g(\gamma, \beta, p)\, D(\gamma, p) \exp\{-[b_* + \mu_* + f(\bar{\phi}^b_{*^*}) - f(\bar{\phi}^b)]\, N^p(\gamma)\} \\
&\quad \times \exp\{\sigma(\operatorname{Int}\gamma \mid \bar{\phi}^b) + \sigma([\operatorname{Int}\gamma]^* \mid \bar{\phi}^b_{*^*})\}
\end{aligned} \tag{2.5.16}$$

Similarly, for γ_* with $N^{d-p}(\gamma_*) \leqslant k+1$, we get

$$\begin{aligned}
\phi^b_{*^*}(\gamma_*(&= \{\exp[-(b_* + \mu_*)\, N^{d-p}(V(\gamma_*))]\} \frac{\mathbf{Z}^{\text{cryst}}(\gamma_*, \beta^*, d-p)}{\mathscr{Z}(\operatorname{Int}\gamma_* \mid \phi^b_{*^*})} \\
&\leqslant g(\gamma_*, \beta^*, d-p(\, D(\gamma_*, d-p) \\
&\quad \times \exp\{-[b+\mu+f(\bar{\phi}^b) - f(\bar{\phi}^b_{*^*})]\, N^{d-p}(\gamma_*)\} \\
&\quad \times \exp\{\sigma(\operatorname{Int}\gamma_* \mid \bar{\phi}^b_{*^*}) + \sigma([\operatorname{Int}\gamma_*]^* \mid \bar{\phi}^b)\}
\end{aligned} \tag{2.5.17}$$

Taking into account that $|g(\gamma, \beta, p)| \leqslant 1$ and $|g(\gamma_*, \beta^*, d-p)| \leqslant 1$ and using the estimates $\mu + b \geqslant \mu = \beta$, $\mu_* + b_* \geqslant \mu_* \geqslant (p/d) \log q$, and (2.5.3), and evaluating

$$|f(\bar{\phi}^b)| \leqslant e^{-\tau} \qquad \text{and} \qquad |f(\bar{\phi}^b_{*^*})| \leqslant e^{-\tau}$$

we finally get

$$|\phi^b(\gamma)| \leqslant e^{-\tau N^p(\gamma)}, \qquad |\phi^b_{*^*}(\gamma_*)| \leqslant e^{-\tau N^{d-p}(\gamma_*)} \tag{2.5.18}$$

Taking $\phi = \phi^b$ and $\phi_* = \phi^b_{*^*}$ thus immediately yields (2.5.7) and (2.5.8) and also (2.5.6), which is actually identical with (2.5.13), since $\phi^b = \bar{\phi}^b$ and $\phi^b_{*^*} = \bar{\phi}^b_{*^*}$.

To prove the second part of the proposition, let us first suppose that $\min(b, b^*) > 0$. Referring to the fact that the Lipschitz constants of $f(\bar{\phi}^\alpha)$ and $f(\bar{\phi}^\alpha_{*^*})$ are at most 1/2 [cf. (2.5.14)], one would then prove that there exists $\varepsilon > 0$ and parameters $\tilde{b}$, $\tilde{b}_*$ such that

$$\tilde{b} + \mu + f(\bar{\phi}^{\tilde{b}}) = \tilde{b}_* + \mu_* + f(\bar{\phi}^{\tilde{b}}_{*^*}) = f_p(\beta H) - \varepsilon$$

Using this equality instead of (2.5.13), one might show in the same way as above that $\phi^{\tilde{b}}$ and $\phi^{\tilde{b}}_{*^*}$ are τ-functionals and thus

$$\tilde{b} + \mu + f(\phi^{\tilde{b}}) = \tilde{b}_* + \mu^* + f(\phi^{\tilde{b}}_{*^*}) < f_p(\beta H)$$

in contradiction with

$$f_p(\beta H) \leqslant \tilde{b} + \mu + f(\phi^{\tilde{b}})$$

following from

$$\mathbf{Z}^{\text{dil}}(V, \beta, p) = \{\exp[\mu N^p(\mathring{V})]\}\, \mathscr{Z}(V \mid \phi^{\bar{b}}, \bar{b})$$
$$\leqslant \{\exp[(\mu + \bar{b})\, N^p(\mathring{V})]\}\, \mathscr{Z}(V \mid \phi^{\bar{b}})$$

Finally, we shall show that, for a fixed q, there exists only one β_t for which $b = b_* = 0$. To see this, we notice that at such a point necessarily $h(\beta) = h_*(\beta)$, where $h(\beta) = \mu + f(\bar{\phi}^{b=0})$ and $h_*(\beta) = \mu_* + f(\bar{\phi}_*^{b_*=0})$. Thus, our aim is to prove that $h(\beta)$ and $h_*(\beta)$ intersect in a single point.

Consider first μ and μ_* as functions of β (cf. Fig. 1). They intersect at

$$\beta_0 = \log \frac{q-1}{q^{-1+p/d} - 1}$$

An important fact is that the slopes of μ and μ_* around β_0 differ significantly:

$$\left.\frac{d\mu}{d\beta}\right|_{\beta=\beta_0} = 1$$

while

$$\left.\frac{d\mu_*}{d\beta}\right|_{\beta=\beta_0} = q^{p/d-1}$$

is very small. Taking into account that $|f(\bar{\phi}^{b=0})|$, $|f(\bar{\phi}_*^{b_*=0})|$ as well as $|(d/d\beta)\, f(\bar{\phi}^{b=0})|$, $|(d/d\beta)\, f(\bar{\phi}_*^{b_*=0})|$ may be bounded by $e^{-\tau}$, one shows that the functions $h(\beta)$ and $h^*(\beta)$ are contained in a tiny strip around $\mu(\beta)$ and $\mu_*(\beta)$, respectively, and have entirely different slopes. Thus one may conclude that they intersect in a single point β_t near β_0.

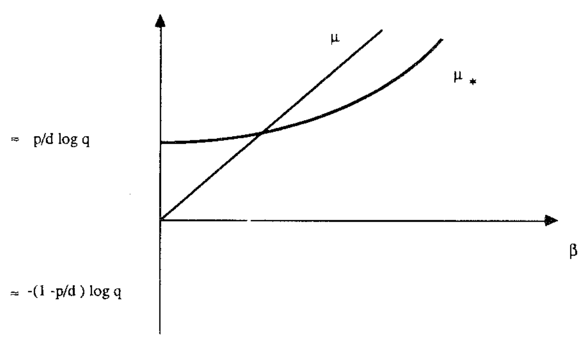

Fig. 1. μ and μ_* as functions of β.

To show the bound on $|(d/d\beta)\, f(\bar{\phi}^{b=0})|$ [and on $|(d/d\beta)\, f(\bar{\phi}_*^{b_*=0})|$], one proceeds in a similar fashion to the proof of (2.5.14).

When estimating $|(d/d\beta)\, f(\phi^{b=0})|$ we again use (2.5.11) with $\alpha=0$ and evaluate $|(d/d\beta) \log \mathscr{Z}(\gamma \mid \phi^{\alpha=0})|$ and $|(d/d\beta) \log \mathscr{Z}(\operatorname{Int} \gamma \mid \phi^{\alpha=0})|$ in terms of mean energies in ensembles corresponding to crystal and diluted partition functions, respectively, with bounds proportional to $N^p(V(\gamma))$, resp. $N^p(\operatorname{Int} \gamma)$. ∎

3. PROOF OF THEOREMS

3.1. Proof of Theorems 1.1 and 1.4

The discontinuity at β_t of the derivative of the free energy with respect to β is associated with the fact that the expectation of $\delta[d\sigma(s^p)]$ takes different values according to the ordered and free boundary conditions (these expectations are respectively right and left derivatives[30]). To introduce these boundary conditions, we let Λ be a rectangular box, $E(\Lambda)$ the envelope of Λ, and V the union of the envelope and the fringe of Λ, $V=E(\Lambda) \cup F(\Lambda)$. The free b.c. are obtained as thermodynamic limit with closed complex $E(\Lambda)$, while the ordered b.c. are obtained with open complex V:

$$\langle \cdot \rangle^{\mathrm{f}}(\beta, p) = \lim_{\Lambda \uparrow \mathbb{Z}^d} \langle \cdot \rangle(E(\Lambda), \beta, p)$$

$$\langle \cdot \rangle^{0}(\beta, p) = \lim_{\Lambda \uparrow \mathbb{Z}^d} \langle \cdot \rangle(V, \beta, p)$$

We define analogously the above b.c. for the dual model. Let us now consider the expectation value

$$\langle \delta[d\sigma(s^p)] \rangle(V, \beta, p) = 1 - \operatorname{Prob}\{d\sigma(s^p)] \neq 0 \ (\operatorname{mod} q) \mid \text{“ord”}\}$$

Since for every configuration $\sigma \in C^{p-1}(\mathbb{L})$ such that $d\sigma(s^p) \neq 0 \ (\operatorname{mod} q)$ and $d\sigma(s^p) = 0 \ (\operatorname{mod} q)$ on $\mathbb{L} \backslash V$ there exits a contour γ belonging to a family of external contours $\theta \subset V$ such that $s^p \in V(\gamma)$, the above probability is, referring to the relations (2.5.7) and (2.5.9), bounded by

$$\sum_{\gamma: s^p \in \gamma} \ \sum_{\substack{\theta \subset V: \\ \gamma \in \theta}} e^{bN^p(V(\theta))} \frac{\mathscr{Z}(\theta \mid \phi^b)}{\mathscr{Z}(V \mid \phi^b, b)}$$

For the dual model we use the relations (2.5.8) and (2.5.10) to get

$$\begin{aligned} &\langle \delta[d\sigma_*(s_*^{d-p})] \rangle([E(\Lambda)]^*, \beta^*, d-p) \\ &\quad \geq 1 - \sum_{\gamma_*: s_*^{d-p} \in \gamma_*} \ \sum_{\substack{\theta_* \subset [E(\Lambda)]^*: \\ \gamma_* \in \theta_*}} e^{b_* N^{d-p}(V(\theta_*))} \frac{\mathscr{Z}(\theta_* \mid \phi_*^{b_*})}{\mathscr{Z}([E(\Lambda)]^* \mid \phi_*^{b_*}, b_*)} \end{aligned}$$

Whenever the inequalities (2.5.4) and (2.5.5) are satisfied, ϕ^b and ϕ^b_* are τ-functionals (cf. Proposition 2.5.1), and the parameter b vanishes for $\beta \geqslant \beta_t$, while b_* vanishes for $\beta \leqslant \beta_t$. Hence, using the standard estimates on probability of contours for contour models, we get

$$\langle \delta[d\sigma(s^p)] \rangle^0(\beta, p) > 1 - O(e^{-\tau}) \qquad \text{whenever } \beta \geqslant \beta_t(q, p, d)$$
$$\langle \delta[d\sigma_*(s_*^{d-p})] \rangle^0(\beta^*, d-p) > 1 - O(e^{-\tau}) \qquad \text{whenever } \beta \leqslant \beta_t$$

Applying the relation (2.2.9), we get

$$\langle \delta[d\sigma(s^p)]^{\mathrm{f}}(\beta, p) < O(e^{-\tau}) + O(q^{1/d}) \qquad \text{whenever} \quad \beta \leqslant \beta_t$$

Since (2.5.4) and (2.5.5) are satisfied for $p=1$, $d \geqslant 2$, and $d=3$, with $\tau \geqslant \text{cst} \log q$, as proved in Appendix B, Theorems 1.1 and 1.4 are implied. We notice that we also get the positivity of the magnetization in Theorem 1.2 with an analogous proof, since for every configuation $\sigma \in C^{p-1}(\mathbb{L})$ such that $\sigma(x_0) \neq 0 \pmod q$ and $\sigma(x) = 0$ if $x \in \mathbb{L} \setminus V$ there exists a contour γ such that $V(\gamma) \supset x_0$.

3.2. Proof of Theorem 1.3

For convenience we shall use the notations introduced in Section 2 to rewrite surface tensions $\tau^{\alpha_1, \alpha_2}(\beta)$ and $\tau^{\alpha, \mathrm{f}}(\beta)$. Namely, we let V be the envelope of $\Lambda_{L,M}$, $V = E(\Lambda_{L,M})$. We define subsets of sites in $B(V)$, $B^+(V)$, and $B^-(V)$ by

$$B^+(V) = \{x \in \mathbb{Z}^d \mid x \in B(V) \cap \mathbb{L}^0,\, x^d \geqslant 0\}$$
$$B^-(V) = \{x \in \mathbb{Z}^d \mid x \in B(V) \cap \mathbb{L}^0,\, x^d < 0\}$$

Let us introduce partition functions

$$\mathbf{Z}(V, \beta, 1 \mid \mathrm{bc}) = \sum_{\sigma \in C^0(V)} e^{-\beta H^1_V(\sigma)} \chi^{\mathrm{bc}}(\sigma)$$

where H is defined by the relation (2.2.2) and $\chi^{\mathrm{bc}}(\sigma)$ characterizes boundary conditions. In terms of these partition functions the surface tensions $\tau^{\alpha_1, \alpha_2}(\beta)$ and $\tau^{\alpha, \mathrm{f}}(\beta)$ may be rewritten as

$$\tau^{\alpha_1, \alpha_2}(\beta) = \operatorname*{Lim}_{L \uparrow \infty; M \uparrow \infty} \frac{1}{L^{d-1}} \log \frac{\mathbf{Z}(V, \beta, 1 \mid \alpha_1, \alpha_2)}{[\mathbf{Z}(V, \beta, 1 \mid \alpha_1)\, \mathbf{Z}(V, \beta, 1 \mid \alpha_2)]^{1/2}}$$
$$\tau^{\alpha, \mathrm{f}}(\beta) = \operatorname*{Lim}_{L \uparrow \infty; M \uparrow \infty} \frac{1}{L^{d-1}} \log \frac{\mathbf{Z}(V, \beta, 1 \mid \alpha, \mathrm{f})}{[\mathbf{Z}(V, \beta, 1 \mid \alpha)\, \mathbf{Z}(V, \beta, 1 \mid \mathrm{f})]^{1/2}}$$

where the corresponding characteristic functions are defined by

$$\chi^{\alpha_1,\alpha_2}(\sigma)=\prod_{x\in B^+(V)}\delta[\sigma(x-\alpha_1)]\prod_{x\in B^-(V)}\delta[\sigma(x-\alpha_2)],$$

$$\chi^{\alpha}(\sigma)=\prod_{x\in B(V)}\delta[\sigma(x-\alpha)]$$

$$\chi^{\alpha,\mathrm{f}}(\sigma)=\prod_{x\in B^+(V)}\delta[\sigma(x-\alpha)],$$

$$\chi^{\mathrm{f}}(\sigma)=1$$

3.2.1. Proof of the Statement (a) of the Theorem. Consider the partition function $\mathbf{Z}(V,\beta,1\,|\,\alpha_1,\alpha_2)$ and observe that for any configuration σ from $C^0(V)$ such that $\chi^{\alpha_1,\alpha_2}(\sigma)=1$, there exist, in V, two disjoint components U (up) and D (down) satisfying the properties:

1. The boundary $B(U)$ of U contains $B^+(V)$ and the boundary $B(D)$ of D contains $B^-(V)$.

2. The lattice sites in $B(U)$ are such that $\sigma(x)=\alpha_1$ while for the lattice sites in $B(D)$ it is $\sigma(x)=\alpha_2$.

3. The links s^1 of the complexes $F(U\cap\mathbb{L}^0)\cap U$ and $F(D\cap\mathbb{L}^0)\cap D$ are disordered $\sigma(\partial s^1)\neq 0 \pmod q$.

We shall write in this section $F(K)$ instead of $F(K\cap\mathbb{L}^0)$. Let us define the subcomplexes γ_1, γ_2, and I by (see Fig. 2)

$$\gamma_1=F(U)\cap V,\qquad \gamma_2=F(D)\cap V,\qquad I=V\backslash(U\cup D\cup\gamma_1\cup\gamma_2)$$

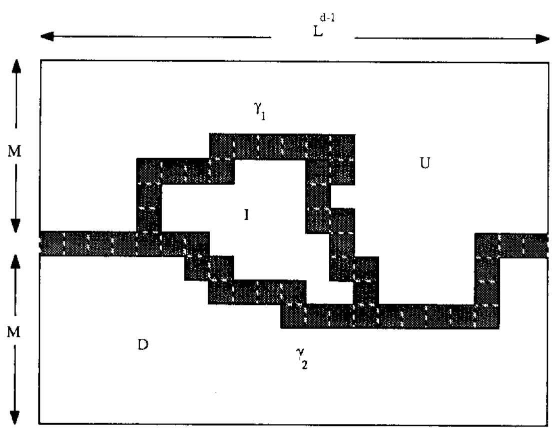

Fig. 2. The contours γ_1 and γ_2 divide V into two "ordered regions" U and D and a "disordered" one I.

We notice that the fringe of I equals the symmetric difference of γ_1 and γ_2. We have

$$F(I) = \gamma_1 \triangle \gamma_2 = \gamma_1 \cup \gamma_2 - \gamma_1 \cap \gamma_2$$

and let

$$V(\gamma_1 \triangle \gamma_2) = I \cup (\gamma_1 \triangle \gamma_2)$$

Thus, the partition function $\mathbf{Z}(V, \beta, 1 \mid \alpha_1, \alpha_2)$ will be written in terms of partition functions defined on the subcomplexes U, D, and $V(\gamma_1 \triangle \gamma_2)$. Namely,

$$\begin{aligned}
&\mathbf{Z}(V, \beta, 1 \mid \alpha_1, \alpha_2) \\
&\quad = \sum_{\gamma_1;\gamma_2} \mathbf{Z}(U, \beta, 1 \mid \alpha_1)\, \mathbf{Z}(D, \beta, 1 \mid \alpha_2)\, \mathbf{Z}(V(\gamma_1 \triangle \gamma_2), \beta, 1 \mid \tilde{\alpha}_1, \tilde{\alpha}_2)
\end{aligned}$$

where

$$\begin{aligned}
&\mathbf{Z}(V(\gamma_1 \triangle \gamma_2), \beta, 1 \mid \tilde{\alpha}_1, \tilde{\alpha}_2) \\
&\quad = \sum_{\sigma \in C^0(I)} e^{-\beta H^1_V(\sigma)} \prod_{\substack{s^1 \in \gamma_1; \\ x \in \partial s^1 \cap B(I)}} \{1 - \delta[\sigma(x - \alpha_1)]\} \\
&\qquad \times \prod_{\substack{s^2 \in \gamma_2; \\ x \in \partial s^2 \cap B(I)}} \{1 - \delta[\sigma(x - \alpha_2)]\}
\end{aligned}$$

It is easy to show that

$$\mathbf{Z}(V(\gamma_1 \triangle \gamma_2), \beta, 1 \mid \tilde{\alpha}_1, \tilde{\alpha}_2) \leqslant \Xi^{\mathrm{g.f.}}(V(\gamma_1 \triangle \gamma_2), \beta, 1 \mid \mathrm{dis})$$

From the symmetry of the Hamiltonian we then deduce

$$\begin{aligned}
&\frac{\mathbf{Z}(V, \beta, 1 \mid \alpha_1, \alpha_2)}{[\mathbf{Z}(V, \beta, 1 \mid \alpha_1)\, \mathbf{Z}(V, \beta, 1 \mid \alpha_2)]^{1/2}} \\
&\quad \leqslant \sum_{\gamma_1;\gamma_2} \frac{\mathbf{Z}(U, \beta, 1 \mid 0)\, \mathbf{Z}(D, \beta, 1 \mid 0)\, \Xi^{\mathrm{g.f.}}(V(\gamma_1 \triangle \gamma_2), \beta, 1 \mid \mathrm{dis})}{\mathbf{Z}(V, \beta, 1 \mid 0)}
\end{aligned}$$

The above partition functions are related to the parametric contour statistical sums defined in Section 2.5 by

$$\mathbf{Z}(V, \beta, 1 \mid 0) = e^{\beta N^1(V)}\, \mathscr{Z}(V \mid \phi^b, b)$$

$$\mathbf{Z}(U, \beta, 1 \mid 0) = e^{\beta N^1(U)}\, \mathscr{Z}(U \mid \phi^b, b)$$

$$\mathbf{Z}(D, \beta, 1 \mid 0) = e^{\beta N^1(D)}\, \mathscr{Z}(D \mid \phi^b, b)$$

$$\Xi^{\mathrm{g.f.}}(V(\gamma_1 \triangle \gamma_2), \beta, 1 \mid \mathrm{dis}) = e^{(b+\beta)\, N^1(V(\gamma_1 \triangle \gamma_2))}\, \phi^b(\gamma_1 \triangle \gamma_2)\, \mathscr{Z}(I \mid \phi^b)$$

For $\beta \geqslant \beta_t(q, p=1, d)$ the parameter b vanishes (cf. Proposition 2.5.1). Thus,

$$\frac{\mathbf{Z}(V, \beta, 1 \mid \alpha_1, \alpha_2)}{[\mathbf{Z}(V, \beta, 1 \mid \alpha_1)\, \mathbf{Z}(V, \beta, 1 \mid \alpha_2)]^{1/2}} \leqslant \sum_{\gamma_1 ; \gamma_2} \phi^b(\gamma_1 \triangle \gamma_2)\, e^{-\beta N^1(\gamma_1 \cap \gamma_2)})$$

Since ϕ is a τ-functional as shown in Appendix B and the contours γ_1 and γ_2 are such that $N^1(\gamma) \geqslant L^{d-1}$, the proof follows in a standard way.

3.2.2. Proof of the statement (b) of the Theorem. We consider the partition function $\mathbf{Z}(V, \beta, 1 \mid 0, \mathrm{f})$. For each configuration σ belonging to $C^0(V)$ and such that $\chi^{0,\mathrm{f}}(\sigma) = 1$ there exists a closed connected component $U \subset V$ such that the boundary $B(U)$ of U contains $B^+(V)$, $B(U) \supset B^+(V)$, and all s^1 in $F(U) \cap V$ are disordered: $\sigma(\partial s^1) \neq 0 \pmod{q}$.

We introduce D as the closed component defined by $D = V \backslash [U \cup (F(U) \cap V)]$ and we denote by $F(D)$ the fringe of D (see Fig. 3). The fringe $F(V)$ of V will be decomposed into subcomplexes $F^+(V)$ and $F^-(V)$, $F(V) = F^+(V) \cup F^-(V)$, where $F^+(V)$ is the intersection of the fringe $F(B^+)$ of $B^+(V)$ with the fringe $F(V)$ of V, $F^+(V) = F(B^+) \cap F(V)$ [and similarly $F^-(V)$].

Introducing the cell complex $\gamma = F(U) \backslash F^+(V)$, we find that the partition function $\mathbf{Z}(V, \beta, 1 \mid 0, \mathrm{f})$ may be bounded by

$$\sum_{\gamma} \mathbf{Z}(U, \beta, 1 \mid 0)\, \mathbf{Z}(D, \beta, 1 \mid \mathrm{f})$$

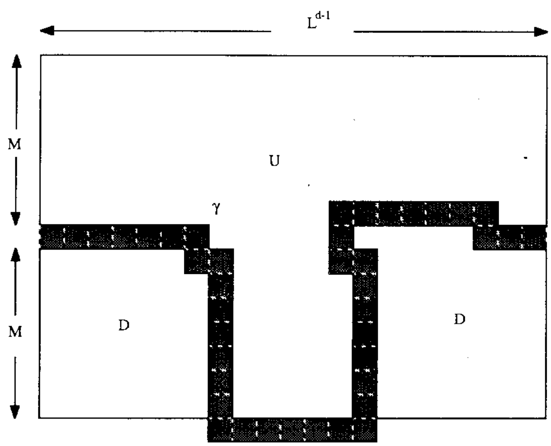

Fig. 3. Contour associated with the interface between an ordered and the free phase.

Now we rewrite this expression in terms of parametric contour statistical sums

$$\mathbf{Z}(U, \beta, 1 \mid 0) = e^{\beta N^1(U)} \mathscr{Z}(U \mid \phi^b, b)$$
$$\mathbf{Z}(V, \beta, 1 \mid 0) = e^{\beta N^1(V)} \mathscr{Z}(V \mid \phi^b, b)$$
$$\mathbf{Z}(D, \beta, 1 \mid \mathrm{f}) = q^{N^1(F(D))/2d} e^{\mu_* N^1(D)} \mathscr{Z}(D^* \mid \phi_*^{b_*}, b_*)$$
$$\mathbf{Z}(V, \beta, 1 \mid \mathrm{f}) = q^{N^1(F(V))/2d} e^{\mu_* N^1(V)} \mathscr{Z}(V^* \mid \phi_*^{b_*}, b_*)$$

The last two identities follow from (2.2.7), (2.4.3), (2.5.9), and Lemma B (Appendix B). Whenever $\beta = \beta_t(q,\ p = 1, d)$, the parameters b and b_* vanish; we then use

$$\frac{\mathscr{Z}(U \mid \phi^b)\, \mathscr{Z}(D^* \mid \phi_*^{b_*})}{[\mathscr{Z}(V \mid \phi^b)\, \mathscr{Z}(V^* \mid \phi_*^{b_*})]^{1/2}} \leqslant \frac{\exp[f(\phi)\, N^1(U) + f(\phi_*)\, N^1(D)]}{\exp[f(\phi)\, N^1(V) + f(\phi_*)\, N^1(V)]} \exp[2N^1(\gamma) e^{-\tau}]$$

which is proved as in ref. 18 and $\beta + f(\phi) = \mu_* + f(\phi_*)$ and $N^1(V) = N^1(U) + N^1(D) + N^1(\gamma \cap F(D))$ to get

$$\frac{\mathbf{Z}(V, \beta, 1 \mid \alpha, \mathrm{f})}{[\mathbf{Z}(V, \beta, 1 \mid \alpha)\, \mathbf{Z}(V, \beta, 1 \mid \mathrm{f})]^{1/2}} \leqslant \sum_{\gamma} q^{-N^1(\gamma \cap F(D))/d + N^1(F(D))/2d - N^1(F(V))/4d}\, e^{2N^1(\gamma) e^{-\tau}} \tag{3.1.1}$$

From the identity

$$\frac{N^1(F(V))}{2} = N^1(F(D)) + N^1(\gamma) - 2N^1(\gamma \cap F(D))$$

we get a bound on the rhs of (3.1.1) by

$$\sum_{\gamma} \exp\left[-\left(\frac{\log q}{2d} - 2e^{-\tau}\right) N^1(\gamma)\right], \qquad \text{where} \quad \tau \geqslant \frac{\log q}{2d}$$

Observing that $N^1(\gamma) \geqslant L^{d-1}$, we conclude the proof of Theorem 1.3.

3.3. Proof of Theorem 1.2

We first consider the statement (b). As seen in Section 3.1, our proofs are based on contour expansions with respect to ordered b.c. Thus, we shall first compute the dual of $\langle q\delta_{\sigma_x, \sigma_y} - 1 \rangle^{\mathrm{f}}(\beta)$ in order to obtain ordered b.c. We shall use the notation of Section 2 and we put $K=E(\Lambda)$. Taking into account that

$$q\delta_{\sigma_x,\sigma_y}-1=\sum_{m=1}^{q-1} e^{(2i\pi/q)m(\sigma_x-\sigma_y)}$$

we obtain, by the procedure of Section 2.2 and using the fact that the homology group $H_1(K)$ is trivial,

$$\begin{aligned}\langle q\delta_{\sigma_x,\sigma_y}-1\rangle^{\mathrm{f}}_{\Lambda}(\beta)&=\langle q\delta_{\sigma_x,\sigma_y}-1\rangle(K,\beta,1)\\&=\sum_{m=1}^{q-1}\frac{\mathbf{Z}_m(K^*,\beta^*,d-1)}{\mathbf{Z}(K^*,\beta^*,d-1)}\end{aligned}\tag{3.3.1}$$

Here the modified partition function $\mathbf{Z}_m(K^*,\beta^*,d-1)$ is defined by

$$\mathbf{Z}_m(K^*,\beta^*,d-1)=\sum_{\sigma\in C^{d-2}(\mathbf{K}^*)}\exp[-\beta^* H^{d-1}_{K^*}(d\sigma+m^*(c))]$$

c is an integral 1-chain such that its boundary consists of x and y. The $d-1$ chaim $*(c)$ is the dual of the chain c: for any $(d-1)$-cell s^{d-1}_* and its dual s^1, $*(c)(s^{d-1}_*)=c(s^1)$.

Since K is closed, K^* is open and we have ordered b.c.

We shall now expand this modified partition function in terms of external contours. Due to the frustrations the family of external contours will always contain a contour γ_* such that $(V(\gamma_*))^*$ contains x and y. More precisely:

For any configuration $\sigma\in C^{d-2}(K^*)$ we say that a cell s^{d-1}_* is ordered if $(d\sigma+m*(c))(d^{d-1}_*)=0$ and that is disordered otherwise. Therefore:

(i) For the d-cell $*(x)=s'^{d}_*$ and $*(y)=s''^{d}_*$ there does not exist a configuration for which all the cells s^{d-1}_* in the boundary $\partial s'^{d}_*$ of s'^{d}_* and in the boundary $\partial s''^{d}_*$ of s''^{d}_* are ordered.

(ii) For the cell s^d_* different from $*(x)$, $*(y)$ there does not exist a configuration σ for which one $(d-1)$-cell of ∂s^d_*) are ordered.

The above two properties are consequences of the relation $dd=0$.

It follows that every configuration σ contains a family of mutually external contours θ_* satisfying:

1. $(d\sigma+m*(c))(s^{d-1}_*)=0$ if $s^{d-1}_*\in\mathrm{Ext}_{K^*}\theta_*$ and $(d\sigma+m*(c))(s^{d-1}_*)\neq 0$ if $s^{d-1}_*\in\theta_*$.

2. In each such family θ_* there exists a unique contour γ_* such that $\{x;y\}\in[V(\gamma_*)]^*$ and for every such contour γ_* there exists an integral

2-chain c_{γ_*} whose boundaries are $*(x)$ and $*(y)$ and all the cells for which $c_{\gamma_*}(s_*^{d-1}) \neq 0$ belong to $V(\gamma_*)$ ($\{c_{\gamma_*}\} \subset V(\gamma_*)$).

Then $\mathbf{Z}_m(K^*, \beta^*, d-1)$ is expanded as

$$\mathbf{Z}_m(K^*, \beta^*, d-1) = \sum_{\substack{\gamma_*: \\ \{x; y\} \in [V(\gamma_*)]^*}} \sum_{\substack{\theta_* \subset K^*: \\ \gamma_* \subset \theta_*}} \sum_{\sigma \in C^{d-2}(K^*)} \chi_m \exp[-\beta^* H_{K^*}^{d-1}(d\sigma + mc_{\gamma_*})]$$

where

$$\chi_m = \prod_{s_*^{d-1} \in \mathrm{Ext}_{K^*}\theta_*} \delta[d\sigma(s_*^{d-1})] \prod_{s_*^{d-1} \in \theta_*} \{1 - \delta[(d\sigma + mc_{\gamma_*})(s_*^{d-1})]\}$$

From the statement (b) of Lemma 2.2.1 we get

$$\begin{aligned} &\mathbf{Z}_m(K^*, \beta^*, d-1) \\ &\quad = \sum_{\substack{\gamma_*: \\ \{x; y\} \in [V(\gamma_*)]^*}} \sum_{\substack{\theta_* \subset K^*: \\ \theta_* \supset \gamma_*}} |Z^{d-2}(K^*)| \exp[\beta^* N^{d-1}(\mathrm{Ext}_{K^*}\theta_*)] \\ &\qquad \times \Xi^{\mathrm{g.f.}}(\theta_* \backslash \gamma_*, \beta^*, d-1 \mid \mathrm{dis})\, \Xi_m^{\mathrm{g.f.}}(\gamma_*, \beta^*, d-1 \mid \mathrm{dis}) \end{aligned} \tag{3.3.2}$$

where

$$\begin{aligned} &\Xi_m^{\mathrm{g.f.}}(\gamma_*, \beta^*, d-1 \mid \mathrm{dis}) \\ &\quad = \sum_{z \in Z^{d-1}(V(\gamma_*))} \{\exp[-\beta^* H_{V(\gamma_*)}^{d-1}(z + mc_{\gamma_*})]\} \\ &\qquad \times \prod_{s_*^{d-1} \in \gamma_*} \{1 - \delta[(z + mc_{\gamma_*})(s_*^{d-1})]\} \end{aligned}$$

We proceed as in the proof of Lemma 2.4.1 to get, from (2.2.7) and (2.2.5),

$$\begin{aligned} &\Xi_m^{\mathrm{g.f.}}(\gamma_*, \beta^*, d-1 \mid \mathrm{dis}) \\ &\quad \leqslant |H^{d-1}(V(\gamma_*))|\, q^{N^{d-2}(\gamma_*)}\, \mathbf{Z}(\mathrm{Int}\,\gamma_*, \beta^*, d-1) \\ &\quad = \left(\frac{e^{\beta^*} - 1}{q}\right)^{N^{d-1}(\mathrm{Int}\,\gamma_*)} |Z^{d-1}(V(\gamma_*))|\, \Xi^{\mathrm{g.f.}}([\mathrm{Int}\,\gamma_*]^*, \beta, 1) \end{aligned}$$

which is up to the term $g(\gamma_*, \beta^*, d-1)$ the rhs of the fourth identity of Lemma 2.4.1. Thus, by defining

$$\mathbf{Z}_m^{\mathrm{cryst}}(\gamma_*, \beta^*, d-1) = [(e^{\beta} - 1) q^{1/d-1}]^{N^{d-1}(V(\gamma_*))}\, \Xi_m^{\mathrm{g.f.}}(\gamma_*, \beta^*, d-1 \mid \mathrm{dis})$$

and referring to the proof of Proposition 5.1 [cf. (2.5.17) and (2.5.18)] we get

$$\begin{aligned}
&\{\exp[-b_* + \beta^*)\, N^{d-1}(V(\gamma_*))]\}\, \frac{\Xi_m^{\text{g.f.}}(\gamma_*, \beta^*, d-1 \mid \text{dis})}{\mathscr{Z}(\operatorname{Int}\gamma_* \mid \phi_*^{b_*})} \\
&\quad = \{\exp[-b_* + \mu_*)\, N^{d-1}(V(\gamma_*))]\}\, \frac{\mathbf{Z}_m^{\text{cryst}}(\gamma_*, \beta^*, d-1)}{\mathscr{Z}(\operatorname{Int}\gamma_* \mid \phi_*^{b_*})} \\
&\quad \leqslant D(\gamma_*, d-p) \exp\{-[b+\mu+f(\phi^b) - (f(\phi_*^{b_*})]\, N^{d-p}(\gamma_*)\} \\
&\qquad \times \exp\{\sigma(\operatorname{Int}\gamma_* \mid \phi_*^{b_*}) + \sigma([\operatorname{Int}\gamma_*]^* \mid \phi^b)\} \\
&\quad \leqslant \exp[-\tau N^{d-1}(\gamma_*)]
\end{aligned} \tag{3.3.3}$$

which, combined with (3.3.2), (2.4.3), and (2.5.8), leads to

$$\begin{aligned}
&\mathbf{Z}_m(K^*, \beta^*, d-1) \\
&\quad \leqslant |Z^{d-2}(K^*)|\, \{\exp[\beta^* N^{d-1}(K^*)]\} \sum_{\substack{\gamma_*: \\ \{x; y\} \in [V(\gamma_*)]^*}} \exp[-\tau N^{d-1}(\gamma_*)] \\
&\qquad \times \sum_{\substack{\theta_* \subset K^*: \\ \theta_* \supset \gamma_*}} \{\exp[b_* N^{d-1}(V(\theta_*))]\}\, \mathscr{Z}(\theta_* \backslash \gamma_* \mid \phi_*^{b_*})\, \mathscr{Z}(\operatorname{Int}\gamma_* \mid \phi_*^{b_*})
\end{aligned}$$

Whenever $\beta \leqslant \beta_t$ $(q, p = 1, d)$, the parameter b_* vanishes; therefore

$$\begin{aligned}
&\mathbf{Z}_m(K^*, \beta^*, d-1) \\
&\quad \leqslant |Z^{d-2}(K^*(|\, \{\exp[\beta^* N^{d-1}(K^*)]\}\, \mathscr{Z}(K^* \mid \phi_*^{b_*}, b_*) \\
&\qquad \times \sum_{\substack{\gamma_*: \\ \{x; y\} \in [V(\gamma_*)]^*}} \exp[-\tau N^{d-1}(\gamma_*)] \\
&\quad = \mathbf{Z}(K^*, \beta^*, d-1) \sum_{\substack{\gamma_*: \\ \{x; y\} \in [V(\gamma_*)]^*}} \exp[-\tau N^{d-1}(\gamma_*)]
\end{aligned}$$

Since the above contours are such that $N^{d-1}(\gamma_*) \geqslant d(x, y)$, referring to (3.3.1), we conclude by standard arguments that the inverse correlation length is strictly positive for $\beta \geqslant \beta_t$. The positivity of the magnetization was proved in Section 3.1. The remaining part of the theorem may be proven in the same way as in ref. 13.

3.4. Proof of Theorem 1.5

According to the notation of Section 2, we introduce the ordered and free b.c. as in Sections 3.1 and 3.3 and define

$$\langle q\delta_{\sigma(\mathscr{L}),0}-1\rangle^{\mathrm{f}}(\beta)=\lim_{\Lambda\uparrow\mathbb{Z}^d}\langle q\delta_{\sigma(\mathscr{L}),0}-1\rangle^{\mathrm{f}}_{\Lambda}(\beta)$$

$$=\lim_{\Lambda\uparrow\mathbb{Z}^d}\langle q\delta_{\sigma(\tilde{\mathscr{L}}),0}-1\rangle(E(\Lambda),\beta,2)$$

$$\langle q\delta_{\sigma(\mathscr{L}),0}-1\rangle^{0}(\beta)=\lim_{\Lambda\uparrow\mathbb{Z}^d}\langle q\delta_{\sigma(\mathscr{L}),0}-1\rangle^{0}_{\Lambda}(\beta)$$

$$=\lim_{\Lambda\uparrow\mathbb{Z}^d}\langle q\delta_{\sigma(\tilde{\mathscr{L}}),0}-1\rangle(E(\Lambda)\cup F(\Lambda),\beta,2)$$

Here Λ is a rectangular box which contains the loop $\mathscr{L}$, and $\tilde{\mathscr{L}}$ denotes the integral 1-cycle, which takes the value 1 on the oriented 1-cells from $\mathscr{L}$ and is zero otherwise (the orientation of a link in $\mathscr{L}$ is clearly one of the two orientations such that $\partial\tilde{\mathscr{L}}=0$).

3.4.1. Proof of Area Law Decay. The area law decay is obtained with free b.c. Hence we shall first compute the dual of the Wilson parameter to obtain ordered b.c. in order to apply a contour expansion. We let $K=E(\Lambda)$ and proceed as in the proof of Theorem 1.2 to get

$$\langle q\delta_{\sigma(\mathscr{L}),0}-1\rangle(K,\beta,2)=\sum_{m=1}^{q-1}\frac{\mathbf{Z}_m(K^*,\beta^*,d-2)}{\mathbf{Z}(K^*,\beta^*,d-2)}$$

where

$$\mathbf{Z}_m(K^*,\beta^*,d-2)=\sum_{\sigma\in C^{d-3}(K^*)}\exp[-\beta^*H^{d-2}_{K^*}(d\sigma+m*(c))]$$

Here c is an integral 2-chain in K whose boundary is $\tilde{\mathscr{L}}$: $\partial c=\mathscr{L}$. The $(d-2)$-chain $*(c)$ is the dual of the chain c: for any $(d-2)$-cell s^{d-2}_* and its dual s^2, $*(c)(s^{d-2}_*)=c(s^2)$.

We shall expand the above modified partition function in terms of external contours. Due to the frustrations, the family of external contours θ_* contains always a contour γ_* such that $[V(\gamma_*)]^*\supset\mathscr{L}$. We proceed now as in Section 3.3. We have

$$\mathbf{Z}_m(K^*,\beta^*,d-2)=\sum_{\substack{\gamma_*\\ \{\mathscr{L}\}^*\subset V(\gamma_*)}}\;\sum_{\substack{\theta_*\subset K^*:\\ \theta_*\supset\gamma_*}}|Z^{d-3}(K^*)|\exp[\beta^*N^{d-2}(\mathrm{Ext}_{K^*}\theta_*)]$$
$$\times\,\Xi^{\mathrm{g.f.}}(\theta_*\backslash\gamma_*,\beta^*,d-2\,|\,\mathrm{dis})\,\Xi^{\mathrm{g.f.}}_m(\gamma_*,\beta^*,d-2\,|\,\mathrm{dis})$$

where the last partition function is defined by

$$\Xi^{\mathrm{g.f.}}_m(\gamma_*,\beta^*,d-2\,|\,\mathrm{dis})$$
$$=\sum_{z\in Z^{d-2}(V(\gamma_*))}\exp[-\beta^*H^{d-2}_{V(\gamma_*)}(z+mc_{\gamma_*})]\prod_{s^{d-2}_*\in\gamma_*}\{1-\delta[(z+mc_{\gamma_*})(s^{d-2}_*)]\}$$

and satisfies

$$\{\exp[-(b_* + \beta^*)\, N^{d-2}(V(\gamma_*))]\}\, \frac{\Xi_m^{\text{g.f.}}(\gamma_*, \beta^*, d-1 \mid \text{dis})}{\mathscr{Z}(\operatorname{Int} \gamma_* \mid \phi_*^{b_*})}$$
$$\leqslant \exp[-\tau N^{d-2}(\gamma_*)]$$

according to the proof of Proposition 2.5.1. Therefore

$$\begin{aligned}
&\mathbf{Z}_m(K^*, \beta^*, d-2) \\
&\quad \leqslant |Z^{d-2}(K^*|)\, e^{\beta^* N^{d-1}(K^*)} \sum_{\substack{\gamma_* \\ \{\mathscr{L}\}^* \subset V(\gamma_*)}} e^{-\tau N^{d-2}(\gamma_*)} \\
&\quad \times \sum_{\substack{\theta_* \subset K^* \\ \gamma_* \subset \theta_*}} e^{b_* N^{d-2}(V(\theta_*))}\, \mathscr{Z}(\theta_* \backslash \gamma_* \mid \phi_*^{b_*})\, \mathscr{Z}(\operatorname{Int} \gamma_* \mid \phi_*^{b_*})
\end{aligned}$$

and for $\beta \leqslant \beta_t(q, p=2, d)$ we then get

$$\mathbf{Z}_m(K^*, \beta^*, d-2) \leqslant \mathbf{Z}(K^*, \beta^*, d-2) \sum_{\substack{\gamma_* \\ \{\mathscr{L}\}^* \subset V(\gamma_*)}} e^{-\tau N^{d-2}(\gamma_*)}$$

Since for all contours occurring in the above sum the number $N^{d-2}(\gamma_*)$ is greater than the number of plaquettes of the minimal surface with boundary $\mathscr{L}$, the proof follows by standard arguments.

3.4.2. Proof of Perimeter Law Decay. Our proof is based on a contour expansion on $\langle q\delta_{\sigma(\mathscr{L}),0} - 1\rangle(K, \beta, 2)$, with $K = E(\Lambda) \cup F(\Lambda)$. It is useful to define the following modified partition function for an open subcomplex V of K:

$$\Xi_\alpha^{\text{g.f.}}(V, \beta, 2) = \sum_{z \in Z^2(V)} e^{(2i\pi/q)\alpha z(c_{|V})}\, e^{-\beta H_V^2(z)} \tag{3.4.1}$$

where c is an integral 2-chain satisfying $\partial c = \tilde{\mathscr{L}}$ and $c_{|V}$ is the restriction to V of c. Let us remark that

$$\langle q\delta_{\sigma(\mathscr{L}),0} - 1\rangle_\Lambda^0(\beta) = \langle q\delta_{\sigma(\tilde{\mathscr{L}}),0} - 1\rangle(K, \beta, 2) = \sum_{\alpha=1}^{q-1} \frac{\Xi_\alpha^{\text{g.f.}}(K, \beta, 2)}{\Xi^{\text{g.f.}}(K, \beta, 2)} \tag{3.4.2}$$

since the cohomology group $H^2(K)$ is trivial.

Lemma 3.4.1. Let V be an open subcomplex of K; then

$$\Xi_\alpha^{\text{g.f.}}(V, \beta, 2) = \sum_{\theta \subset V} e^{\beta N^2(\operatorname{Ext}_V \theta)}\, \Xi_\alpha^{\text{g.f.}}(\theta, \beta, 2 \mid \text{dis})$$

where the sum is over all supports of families of mutually external contours satisfying $V(\theta) \cap \mathbb{L}^1 \subset V$ and

$$\begin{aligned} &\Xi_\alpha^{\text{g.f.}}(\theta, \beta, 2 \mid \text{dis}) \\ &= \sum_{z \in Z^2(V(\theta))} e^{-\beta H^2_{V(\theta)}(z)}\, e^{(2i\pi/q)\alpha z(c_{|V(\theta)})} \prod_{s^2 \in \theta} [1 - \delta(z(s^2))] \\ &= \prod_{\gamma \in \theta} \Xi_\alpha^{\text{g.f.}}(\gamma, \beta, 2 \mid \text{dis}) \end{aligned}$$

Proof. We define $\chi^p_{\theta;K}$ as in the proof of Lemma 2.4.1; thus

$$\left\{ \prod_{s^2 \in K \setminus V} \delta[d\sigma(s^2)]\, e^{(2i\pi/q)\alpha d\sigma(c)} \right\} (K, \beta, 2)$$

$$= \sum_{\theta \subset V} [\chi^2_{\theta;K}\, e^{(2i\pi/q)\alpha d\sigma(c)}](K, \beta, 2)$$

We use the statement (b) of Lemma 2.2.1 to deduce

$$\left\{ \prod_{s^2 \in K \setminus V} \delta[d\sigma(s^2)]\, e^{(2i\pi/q)\alpha d\sigma(c)} \right\} (K, \beta, 2) = |Z^1(K)|\, e^{\beta N^2(K \setminus V)}\, \Xi_\alpha^{\text{g.f.}}(V, \beta, 2)$$

$$[\chi^2_{\theta;K} e^{(2i\pi/q)\alpha d\sigma(c)}](K, \beta, 2) = |Z^1(K)|\, e^{\beta N^2(\text{Ext}_K \theta)}\, \Xi_\alpha^{\text{g.f.}}(\theta, \beta, 2 \mid \text{dis})$$

and we then derive the result. ∎

The family θ of external contours can be divided into two subfamilies:

1. $\gamma \in \underline{\theta} \Leftrightarrow V(\gamma) \cap \mathscr{L} \neq \varnothing$, or for each c such that $\partial c = \tilde{\mathscr{L}}$, there exists $s^2 \in V(\gamma)$ with $c(s^2) = 1$.

2. $\gamma \in \underline{\underline{\theta}} \Leftrightarrow V(\gamma) \cap \{\mathscr{L}\} = \varnothing$, and there exists c such that $\partial c = \tilde{\mathscr{L}}$ and for each $s^2 \in V(\gamma)$, one has $c(s^2) = 0$.

The perimeter law will be a consequence of the following result.

Lemma 3.4.2. Assume $\beta \geqslant \beta_t(q, p = 2, d)$ and let

$$\psi_\alpha(\gamma) = \frac{\{\exp[-\beta N^2(V(\gamma))]\}\, \Xi_\alpha^{\text{g.f.}}(\gamma, \beta_t, 2 \mid \text{dis})}{\{\exp[-\beta N^2(\mathring{\text{Int}}\,\gamma)]\}\, \Xi_\alpha^{\text{g.f.}}(\mathring{\text{Int}}\,\gamma, \beta, 2)}$$

Then

(a) $\quad \{\exp[-\beta N^2(V)]\}\, \Xi_\alpha^{\text{g.f.}}(V, \beta, 2) = \sum_{\partial \in D(V)} \prod_{\gamma \in \partial} \psi_\alpha(\gamma)$

(b) $\quad \psi_\alpha(\gamma) = \phi(\gamma) \qquad \text{if} \quad \gamma \in \underline{\underline{\theta}}$

(c) $\quad |\psi_\alpha(\gamma)| \leqslant \phi(\gamma) \qquad \text{if} \quad \gamma \in \underline{\theta}$

Proof. To prove statement (a), we use the Lemma 3.4.1 and iterate it on $\Xi_{\alpha}^{\text{g.f.}}(\mathring{\text{Int}}\,\gamma, \beta, 2)$.

Whenever $\beta \geqslant \beta t$, ϕ satisfies

$$\phi(\gamma) = \frac{\{\exp[-\beta N^2(V(\gamma))]\}\, \Xi^{\text{g.f.}}(\gamma, \beta, 2 \,|\, \text{dis})}{\{\exp[-\beta N^2(\mathring{\text{Int}}\,\gamma)]\}\, \Xi^{\text{g.f.}}(\mathring{\text{Int}}\,\gamma, \beta, 2)}$$

according to Proposition 2.5.1. Since for any $\gamma \in \underline{\theta}$ one has

$$\Xi_{\alpha}^{\text{g.f.}}(\gamma, \beta, 2 \,|\, \text{dis}) = \Xi^{\text{g.f.}}(\gamma, \beta, 2 \mid \text{dis})$$
$$\Xi_{\alpha}^{\text{g.f.}}(\mathring{\text{Int}}\,\gamma, \beta, 2) = \Xi^{\text{g.f.}}(\mathring{\text{Int}}\,\gamma, \beta, 2)$$

statement (b) follows.

The proof of statement (c) is analogous to the proof of statement c of Proposition 6.1 in ref. 14. ∎

The relation (3.4.2) combined with Lemma 3.4.2 and Proposition 2.5.1 leads to

$$\langle q\delta_{\sigma(\mathscr{L}),0} - 1 \rangle_{\Lambda}^{0}(\beta) = \sum_{\alpha=1}^{q-1} \frac{\sum_{\partial \in D(K)} \prod_{\gamma \in \partial} \psi_{\alpha}(\gamma)}{\sum_{\partial \in D(K)} \prod_{\gamma \in \partial} \phi(\gamma)}$$

Applying the cluster expansion[24,33,34] to both the numerator and the denominator and observing that the corresponding truncated contours functions $\phi^T(C)$ and $\psi_{\alpha}^T(C)$ coincide for clusters C consisting only of contours from $\underline{\theta}$, we get the bound (c) of Theorem 1.5.

The statement (b) is then a consequence of the following inequality[30]:

$$\lim_{\varepsilon \to 0} \left\langle \cos \frac{2\pi}{q} m\sigma(\mathscr{L}) \right\rangle^{\text{f}} (\beta + \varepsilon) \geqslant \left\langle \cos \frac{2\pi}{q} m\sigma(\mathscr{L}) \right\rangle^{0} (\beta), \qquad \varepsilon \geqslant 0$$

since

$$q\delta_{\sigma(\mathscr{L}),0} - 1 = \sum_{m=1}^{q-1} \cos \frac{2\pi}{q} m\sigma(\mathscr{L})$$

This ends the proof of the theorems.

APPENDIX A. CELL COMPLEX FORMALISM

A.1. General Definitions

The cell complex formalism is very efficient in dealing with topological problems inherent to the q-states Potts gauge model. We first introduce it in an abstract sense along the line of ref. 26 (cf. also refs. 35 and 36), and then consider as a particular example a hypercubic lattice $\mathbb{Z}^d$.

A *cell complex* K is a set whose elements are called *cells* such that:

1. A nonnegative integer called dimension is assigned to each cell. The upper bound of the dimensions of all cells is called the dimension of the complex.
2. To each cell s^p of dimension p (a p-cell) there corresponds another p-cell $(-s^p)$ of the same dimension and called the cell of opposite orientation.
3. An integer $I(s^p; s^{p-1})$ called the *incidence number* is assigned to all pairs of cells (s^p, s^{p-1}) in a such way that

$$I(-s^p; s^{p-1}) = I(s^p; -s^{p-1}) = -I(s^p; s^{p-1})$$

A cell complex is called an *a-complex* if

$$\sum_{s^{p-1}} I(s^p; s^{p-1})\, I(s^{p-1}; s^{p-2}) = 0 \tag{A.1}$$

for any two cells s^p and s^{p-2} belonging to K. Another terminology is sometimes used: a cell space instead of a cell complex and a cell complex instead of an a-complex; the above have been introduced in ref. 26.

An *integral p-chain* c^p on the complex K is an odd function on p-cells with values in $\mathbb{Z}$, the group of integers:

$$c^p: \qquad s^p \in K \to c^p(s^p) \in \mathbb{Z}$$

The set of all p-chains over K form an Abelian group denoted by $C^p(K)$. The rank of this group is denoted $N^p(K)$; $2N^p(K)$ is the number of p-cells of K.

A *monomial chain* $m \cdot s^p$ is a chain that takes a value m on s^p and vanishes on all p-cells different from s^p. Hence, any integral chain may be written as a sum of monomial chains:

$$c^p = \sum_i m_i \cdot s_i^p, \qquad 1 \leqslant i \leqslant N^p(K), \quad m_i = c^p(s_i^p)$$

Hereafter s^p denotes either the cell s^p or the monomial chain $1 \cdot s^p$.

One may introduce the *scalar product*

$$(c_1^p, c_2^p) = \sum_i c_1^p(s_i^p)\, c_2^p(s_i^n)$$

and the *boundary* ∂: $C^p(K) \to C^{p-1}(K)$ and the *coboundary* ∂^*: $C^p(K) \to C^{p+1}(K)$ operators defined by

$$\begin{aligned}
\partial s^p &= \sum_j I(s^p; s_j^{p-1}) s_j^{p-1} \\
\partial c^p &= \sum_i m_i \cdot \partial s_i^p = \sum_{i;j} m_i I(s_i^p; s_j^{p-1})\, s_j^{p-1} \\
\partial^* s^p &= \sum_j I(s_j^{p+1}; s^p) s_j^{p+1} \\
\partial^* c^p &= \sum_i m_i \cdot \partial s_i^p = \sum_{i;j} m_i I(s_j^{p+1}; s_i^p)\, s_j^{p+1}
\end{aligned} \tag{A.2}$$

Notice that

$$I(s^p; s^{p-1}) = (\partial s^p, s^{p-1}) = (s^{p,} \partial^* s^{p-1})$$

i.e., ∂^* is the adjoint of ∂ with respect to the scalar product:

$$(\partial c^p, c^{p-1}) = (c^p, \partial^* c^{p-1})$$

A cell complex K_0 is said to be a *cell subcomplex* of the complex K if every element of K_0 is an element of K, every two cells s^p and s^{p+1} have the same incidence number in K as they do in K_0, and every pair of opposites in K_0 is a pair of opposites in K. A cell complex K_0 is said to be closed (respectively open) if it contains with every cell also the cells on its boundary (respectively coboundary). We denote by $\bar{K}_0$ the *closure* of K_0, i.e., the minimal closed cell-complex containing K_0. A complex is said to be *connected* if it cannot be expressed as the union of two nonempty, disjoint, closed subcomplexes.

A hypercubic lattice $\mathbb{Z}^d$ may be considered as a cell complex denoted $\mathbb{L}$. Its 0-cells are vertices, its 1-cells are links, its 2-cells are plaquettes, etc. We shall denote by $\mathbb{L}^p$, $p = 0, 1,..., d$, the set of p-cells in $\mathbb{L}$. The orientation is the usual one and the incidence number $I(s^p; s^{p-1})$ takes values ± 1 if s^{p-1} belongs to the boundary of s^p with respect to the relative orientation and the value 0 otherwise.

Let us consider a cell subcomplex K of $\mathbb{L}$ and restrict the incidence function to K (note that the boundary operation, then, does not coincide with the same operation in $\mathbb{L}$). K will be an a-complex if it satisfies (A.1). In particular, closed and open subcomplexes are a-complexes. Hereafter we shall only consider a-complexes.

The group $C^p(K)$ has two distinguished subgroups with respect to the operator ∂: the group of *p-cycles* $Z_p(K) = \{c^p \mid \partial c^p = 0\}$ and the group of *p-boundaries* $B_p(K) = \{c^p \mid c^p = \partial c^{p+1}\}$. Since $\partial\partial = 0$ as follows from (A.1), every boundary is a cycle: $B_p(K) \subset Z_p(K)$. The converse is not true in general. The factor group $H_p(K) \equiv Z_p(K)/B_p(K)$ is called the p-homology group of the complex K. The rank of $H_p(K)$, denoted by $\pi^p(K)$, is a topological invariant called the pth Betti number. For ∂^* one defines similarly the groups of p-cocycles $Z^p(K)$ and p-coboundaries $B^p(K)$ and the p-cohomology group $H^p(K)$ of a complex K. The Betti number $\pi^p(K)$ characterizes the number of independent p-dimensional holes in K. The other topological invariants are torsion numbers $\tau^p(K)$; they are associated to p-chains c^p which are not boundaries of $(p+1)$-chains in K, whereas $\theta^p c^p$ is a boundary; the integers $\theta_i^p(K)$, $i = 1, 2,..., \tau^p(K)$, are also topological invariants called p-torsion coefficients. τ^p in the number of p-torsion coefficients (a characteristic example is the Klein bottle: $\tau^1 = 1$ and $\theta^1 = 2$).

The following notations will later serve to describe *configurations of lattice models.* A homomorphism σ^p from $C^p(K)$ into an Abelian group G is called a G-valued p-chain. The set of G-valued p-chains of a complex K forms an Abelian group denoted $C^p(K, G)$; in particular, $C^p(K, \mathbb{Z}) = C^p(K)$. Any σ^p is determined by its values on the chains $1 \cdot s^p$, i.e., on the cells s^p; it thus *defines an odd function on the complex K with values in G.*

One may define the differential

$$d\colon \quad C^p(K, G) \to C^{p+1}(K, G)$$

and the codifferential

$$d^*\colon \quad C^p(K, G) \to C^{p-1}(K, G)$$

operators

$$d\sigma^p(c^{p+1}) = \sigma^p(\partial c^{p+1}), \qquad d^*(\sigma^p(c^{p-1}) = \sigma^p(\partial^* c^{p-1}) \tag{A.3}$$

In particular,

$$d\sigma^p(s^{p+1}) = \sum_j I(s^{p+1}; s_j^p)\, \sigma^p(s_j^p)$$

$$d^*\sigma^p(s^{p-1}) = \sum_j I(s_j^p; s^{p-1})\, \sigma^p(s_j^p)$$

Whenever G is a ring with unity, every G-valued p-chain s belonging to $C^p(K, G)$ has a unique decomposition on the cell basis: $\sigma = \sum_i \alpha_i s_i^p$, here α_i belongs to G and $\sum$ denotes the group law of G.

We introduce

1. The group of G-valued p-cycles of K,

$$Z_p(K, G) = \{\sigma^p \in C^p(K, G) \mid d^*\sigma^p = 0\}$$

(here 0 denotes the unit element of G).

2. The group of G-valued p-boundaries of K,

$$B_p(K, G) = \{\sigma^p \in C^p(K, G) \mid \sigma^p = d^*\sigma^{p+1}, \sigma^{p+1} \in C^{p+1}(K, G)\}$$

3. The group of G-valued p-cocycles of K,

$$Z^p(K, G) = \{\sigma^p \in C^p(K, G) \mid d\sigma^p = 0\}$$

4. The group of G-valued p-coboundaries of K,

$$B^p(K, G) = \{\sigma^p \in C^p(K, G) \mid \sigma^p = d\sigma^{p-1}, \sigma^{p-1} \in C^{p-1}(K, G)\}$$

The factor groups $H_p(K, G) = Z_p(K, G)/B_p(K, G)$ and $H^p(K, G) = Z^p(K, G)/B^p(K, G)$ are respectively the G-valued p-homology and the G-valued p-cohomology groups of K.

A.2. Dual Lattice and Dual Complex

Let K be a d-dimensional cell complex; K^* is said to be the dual complex of K if there is a one-to-one correspondence

$$s^p \to *(s^p) = s_*^{d-p} \tag{A.4}$$

between p-cells s^p of K and the $(d-p)$-cells s_*^{d-p} of K^* such that the incidence numbers satisfy the relation

$$I(s^p; s^{p-1}) = I(s_*^{d-p+1}; s_*^{d-p})$$

The lattice

$$(\mathbb{Z}^d)^* = \{x_* \mid x_* = (x^1 + \tfrac{1}{2}, ..., x^i + \tfrac{1}{2}, ..., x^d + \tfrac{1}{2}),\ x^i \in \mathbb{Z}\}$$

is the dual lattice of $\mathbb{Z}^d$. Let $\mathbb{L}^*$ be the complex associated with $(\mathbb{Z}^d)^*$. The complex $\mathbb{L}^*$ is the dual complex of $\mathbb{L}$. For any cell subcomplex K of $\mathbb{L}$ there is a dual complex K^* which is a subcomplex of $\mathbb{L}^*$; if K is closed, K^* is open; if K is open, K^* is closed.

We introduce the operation $*$ (Hodge operation) mapping $C^p(K)$ into $C^{d-p}(K^*)$ and $C^p(K, G)$ into $C^{d-p}(K^*, G)$ by

$$\begin{aligned} *:\quad & c^p \to *(c^p) = c_*^{d-p}, \qquad c_*^{d-p}(s_*^{d-p}) = c^p(s^p) \\ *:\quad & \sigma^p \to *(\sigma^p) = \sigma_*^{d-p}, \qquad \sigma_*^{d-p}(s_*^{d-p}) = \sigma^p(s^p) \end{aligned} \tag{A.5}$$

It follows from (A.3) and (A.5) that

$$*(d\sigma^p) = d^* *(\sigma^p), \qquad *(d^*\sigma^p) = d *(\sigma^p) \tag{A.6}$$

and for integral chains

$$*(\partial c^p) = \partial^* * (c^p), \qquad *(\partial^* c^p) = \partial * (c^p)$$

Therefore the mapping $*$ determines an isomorphism between:

1. The group of G-valued p-cycles of K, $Z_p(K, G)$, and the group of G-valued $(d-p)$-cocycles of K^*, $Z^{d-p}(K^*, G)$.
2. The group of G-valued p-boundaries of K, $B_p(K, G)$, and the group of G-valued $(d-p)$-coboundaries of K^*, $B^{d-p}(K^*, G)$.
3. The G-valued p-homology group of K, $H_p(K, G)$, and the G-valued $(d-p)$-cohomology group of K^*, $H^{d-p}(K^*, G)$.

A.3. ∂-Basis and ∂^*-Basis

A standard result of algebraic topology is that the group $C^p(K)$ admits canonical ∂-basis and ∂^*-basis. A ∂-basis consists in five families of integral p-chains:

$$\{a_i^p \mid i = 1 \cdots \nu^{p-1}\}, \qquad \{x_i^p \mid i = 1 \cdots \tau^{p-1}\}, \qquad \{h_i^p \mid i = 1 \cdots \nu^p\}$$

$$\{b_i^p \mid i = 1 \cdots \nu^p\}, \qquad \{t_i^p \mid i = 1 \cdots \tau^p\}$$

satisfying

$$\begin{aligned}
&N^p = \nu^{p-1} + \tau^{p-1} + \pi^p + \nu^p + \tau^p && \\
&\partial a_i^p = b_i^{p-1}, && i = 1 \cdots \nu^{p-1} \\
&\partial x_i^p = \theta_i^{p-1} t_i^{p-1}, && i = 1 \cdots \tau^{p-1} \\
&\partial h_i^p = 0, && i = 1 \cdots \pi^p \\
&b_i^p = \partial a_i^{p+1}, && i = 1 \cdots \nu^p \\
&\theta_i^p t_i^p = \partial x_i^{p+1}, && i = 1 \cdots \tau^p \\
&\theta_i^p = 0 \; (\mathrm{mod}\; \theta_{i+1}^p) &&
\end{aligned}$$

A ∂^*-basis consists of five families of integral p-chains:

$$\{\bar{a}_i^p \mid i = 1 \cdots \nu^p\}, \qquad \{\bar{x}_i^p \mid i = 1 \cdots \tau^p\}, \qquad \{\bar{h}_i^p \mid i = 1 \cdots \pi^p\}$$

$$\{\bar{b}_i^p \mid i = 1 \cdots \nu^{p-1}\}, \qquad \{\bar{t}_i^p \mid i = 1 \cdots \tau^{p-1}\}$$

satisfying

$$N^p = \nu^{p-1} + \tau^{p-1} + \pi^p + \nu^p + \tau^p$$

$$\partial^*\bar{a}_i^p = \bar{b}_i^{p+1}, \qquad i = 1 \cdots \nu^p$$

$$\partial^*\bar{x}_i^p = \theta_i^{p-1}\bar{t}_i^{p+1}, \qquad i = 1 \cdots \tau^p$$

$$\partial^*\bar{h}_i^p = 0 \qquad i = 1 \cdots \pi^p$$

$$\bar{b}_i^p = \partial^*\bar{a}_i^{p-1} \qquad i = 1 \cdots \nu^{p-1}$$

$$\theta_i^p\bar{t}_i^p = \partial^*\bar{x}_i^{p-1} \qquad i = 1 \cdots \tau^{p-1}$$

We refer to refs. 26 and 36 for suitable examples for our purposes.

The mapping (A.4) sends a ∂-basis into a ∂^*-basis and conversely, and

$$N^p(K) = N^{d-p}(K^*), \qquad \pi^p(K) = \pi^{d-p}(K^*) \qquad \text{for} \quad 0 \leqslant p \leqslant d$$

$$\tau^{p-1}(K) = \tau^{d-p}(K^*) \qquad \text{for} \quad 1 \leqslant p \leqslant d$$

Every p-cycle z belonging to the group $Z_p(K, G)$ has a unique decomposition on a ∂-basis,

$$z = \sum_{i=1}^{\tau^{p-1}} \xi_i x_i^p + \sum_{i=1}^{\pi^p} \mu_i h_i^p + \sum_{i=1}^{\nu^p} \beta_i b_i^p + \sum_{i=1}^{\tau^p} \gamma_i t_i^p$$

Here μ_i, β_i, γ_i belong to G and ξ_i belongs to the group $G(\theta_i^{p-1}) = \{g \in G \mid \theta_i^{p-1}g = 0\}$.

Every p-boundary $b \in B_p(K, G)$ has a unique decomposition on a ∂-basis,

$$b = \sum_{i=1}^{\nu^p} \beta_i b_i^p + \sum_{i=1}^{\tau^p} \rho_i t_i^p$$

Here β_i belong to G and ρ_i belongs to the group $\theta_i^p G = \{\theta_i^p g/g \in G\}$.

Every p-cocycle $z' \in Z^p(K, G)$ has a unique decomposition on a ∂^*-basis:

$$z' = \sum_{i=1}^{\tau^p} \xi_i' \bar{x}_i^p + \sum_{i=1}^{\pi^p} \mu_i' \bar{h}_i^p + \sum_{i=1}^{\nu^{p-1}} \beta_i' \bar{b}_i^p + \sum_{i=1}^{\tau^{p-1}} \gamma_i' \bar{t}_i^p$$

Here μ_i', β_i', γ_i' belong to G and ξ_i' belongs to the group $G(\theta_i^p)$.

Every p-coboundary $b' \in B^p(K, G)$ has a unique decomposition on a ∂^*-basis:

$$b' = \sum_{i=1}^{\nu^{p-1}} \beta_i' \bar{b}_i^p + \sum_{i=1}^{\tau^{p-1}} \gamma_i' \bar{t}_i^p$$

Here β_i' belongs to G and γ_i' belongs to the group $\theta_i^{p-1}G$.

For the above groups the following decomposition in direct sums holds true (see ref. 26 for details):

$$
\begin{aligned}
Z_p(K, G) &\cong \sum_{i=1}^{\tau^{p-1}} G(\theta_i^{p-1}) \oplus \sum_{i=1}^{\pi^p} G \oplus \sum_{i=1}^{\nu^p} G \oplus \sum_{i=1}^{\tau^p} G \\
B_p(K, G) &\cong \sum_{i=1}^{\nu^p} G \oplus \sum_{i=1}^{\tau^p} \theta_i^p G \\
H_p(K, G) &\cong \sum_{i=1}^{\tau^{p-1}} G(\theta_i^{p-1)}) \oplus \sum_{i=1}^{\pi^p} G \oplus \sum_{i=1}^{\tau^p} G/\theta_i^p G \\
Z^p(K, G) &\cong \sum_{i=1}^{\tau^{p-1}} G \oplus \sum_{i=1}^{\nu^{p-1}} G \oplus \sum_{i=1}^{\pi^p} G \oplus \sum_{i=1}^{\tau p} G(\theta_i^p) \\
B^p(K, G) &\cong \sum_{i=1}^{\tau^{p-1}} \theta_i^{p-1} G \oplus \sum_{i=1}^{\nu^{p-1}} G \\
H^p(K, G) &\cong \sum_{i=1}^{\tau^{p-1}} G/\theta_i^{p-1} G \oplus \sum_{i=1}^{\pi^p} G \oplus \sum_{i=1}^{\tau^p} G(\theta_i^p) \\
N^p &= \nu^{p-1} + \tau^{p-1} + \pi^p + \nu^p + \tau^p
\end{aligned}
\tag{A.7}
$$

where the symbols $\cong$ and $\oplus$ denote, respectively, isomorphism and direct sum of groups.

Finally we recall the Alexander duality theorem; we refer the reader to ref. 26 (Vol. 3, pp. 41–42).

Theorem A.1. Let K denote a closed subcomplex of the lattice cell-complex $\mathbb{L}$; then:

(a) For p such that $1 \leqslant p \leqslant d-2$,

$$
\begin{aligned}
H^p(K, G) &\cong H_{d-p-1}([\mathbb{L} \backslash K]^*, G) \\
\pi^p(K) &= \pi^{d-p}([\mathbb{L} \backslash K]^*) \\
\pi^0([\mathbb{L} \backslash K]^*) - 1 &= \pi^{d-1}(K)
\end{aligned}
$$

(b) For p such that $0 \leqslant p \leqslant d-2$,

$$\tau^0(K) = \tau^{d-p-2}([\mathbb{L} \backslash K]^*)$$

(c) K and $[\mathbb{L} \backslash K]^*$ are p-torsion free, $\tau^p = 0$, for $p = 0$ and $p \geqslant d-2$.

APPENDIX B

B.1. Verification of the Generalized Peierls Conditions

To verify (2.5.4) and (2.5.5) from Proposition 2.5.1, we put

$$d(\gamma, p) = q^{-(p/d) N^p(V(\gamma))} \, |Z^p(V(\gamma))|$$
$$d(\gamma_*, p) = q^{-[(d-p)/d] N^{d-p}(V)\gamma_*))} \, |Z^{d-p}(V(\gamma_*))|$$

and evaluate it in the following cases

(a) If $p=1$ and $d \geqslant 2$, we get from the relations (A.7)

$$|Z^1(V(\gamma))| = |Z_{d-1}([V(\gamma)]^*)|$$
$$= q^{N^d([V(\gamma)]^*)} = q^{N^0(\mathrm{Int}\,\gamma)}$$
$$|Z^{d-1}(V(\gamma_*))| = |Z_1([V(\gamma_*)]^*)|$$
$$= q^{N^1([V(\gamma_*)]^*) - N^0([V(\gamma_*)]^*) + \pi^0([V(\gamma_*)]^*)}$$

We have used the fact that closed complexes are 0, $d-2$, $d-1$, and d torsion-free and that $\pi^{d-1}([V(\gamma)]^*) = 0$. Thus,

$$d(\gamma, 1) = q^{N^0(\mathrm{Int}\,\gamma) - N^1(\mathrm{Int}\,\gamma)/d - N^1(\gamma)/d}$$
$$d(\gamma_*, 1) = q^{N^1([V(\gamma_*)]^*/d - N^0([V(\gamma_*)]^*) + \pi^0([V(\gamma_*)]^*)} \tag{B.1}$$

(b) If $p=2$ and $d=3$ we get, proceeding as above,

$$|Z^2(V(\gamma))| = |Z_1([V(\gamma)]^*)|$$
$$= q^{N^1([V(\gamma)]^*) - N^0([V(\gamma)]^*) + \pi^0([V(\gamma)]^*)}$$
$$|Z^2(V(\gamma_*))| = |Z_1([V(\gamma_*)]^*)| = q^{N^0(\mathrm{Int}\,\gamma_*)}$$

Thus

$$d(\gamma, 2) = q^{N^1([V(\gamma)]^*)/3 - N^0([V(\gamma)]^*) + \pi^0([V(\gamma)]^*)}$$
$$d(\gamma_*, 2) = q^{N^0(\mathrm{Int}\,\gamma_*) - N^1(\mathrm{Int}\,\gamma_*)/3 - N^1(\gamma_*)/3} \tag{B.2}$$

To evaluate the expressions in (B.1) and (B.2), we consider a complex $D^{r-1} \subset \mathbb{L}^{r-1}$, $1 \leqslant r \leqslant d$, and introduce its envelope $E(D^{r-1}) = E$ and its fringe $F(D^{r-1}) = F$. We denote by $I^{(l)}(E, x)$ the number of positively oriented 1-cells, $l \in E$, contained in the coboundary $\partial^* x$ of a site, $x \in E$. The number $N^1(F)$ of links in the fringe F of D^{r-1} may be written as a sum

$$N^1(F) = N^1_1(F) + N^1_2(F)$$

with $N_1^1(F)$ the number of links in F such that each link is contained in the coboundary of exactly one site x belonging to E, and $N_2^1(F)$ is the number of links in F such that each link is contained in the coboundary of two sites x belonging to E.

Lemma B.

(a) $$N^0(E) - \frac{N^1(E)}{d} = \frac{N^1(F)}{2d} + \frac{N_2^1(F)}{2d}$$

(b) $$\pi^0(E) \leqslant \sum_{I^{(l)}(E,x) \leqslant d} 2^{-I^{(l)}(E,x)}$$

Proof. Observing that

$$N^1(E) = \sum_{x \in E} \frac{I^{(l)}(E, x)}{2}$$

we get

$$N^0(E) - \frac{N^1(E)}{d} = \frac{1}{2d} \sum_{x \in E} [2d - I^{(l)}(E, x)] \tag{B.3}$$

Taking into account that the maximal number of links $l \in E$ occurring in the coboundary of $x \in E$ is $2d$, we deduce that $2d - I^{(l)}(E, x)$ is the number of links in F contained in the coboundary of x; it follows that

$$\frac{1}{2d} \sum_{x \in E} [2d - I^{(l)}(E, x)] = \frac{N^1(F)}{2d} + \frac{N_2^1(F)}{2d}$$

Note that if $r = 1$, then $N_2^1(F(D^0)) = 0$ by construction of $E(D^0)$.

The bound in (b) on the number of connected components follows if we notice that every p-cell $s^p \in E$ contains 2^p sites of E and that, in a d-dimensional cell complex E, d kinds of connected components can occur, namely, components which are trees, components containing loops, components containing closed surfaces of dimension $(d-1)$. ∎

Now we apply Lemma B to expressions (B.1) and (B.2), respectively. We get

$$d(\gamma, 1) = q^{-N^1(\gamma)/2d} \tag{B.4}$$

because Int γ is of the form $E(D^0)$, and $F(\text{Int}\,\gamma) = \gamma$, by construction of γ. The bound

$$d(\gamma_*, 1) = q^{-N^1(\gamma_*)/2d^2} \tag{B.5}$$

follows by applying the relation (B.3) and the statement (b) of Lemma B to the case where $E=[V(\gamma_*)]^*$ and taking into account that $I^{(l)}([V(\gamma_*)]^*, x)\geqslant 2$ by construction of a contour γ_*.

To get a bound on (B.2), one proceeds as in (B.1) with $d=3$, to obtain

$$d(\gamma, 2)=q^{-N^1(\gamma)/18} \qquad \text{and} \qquad d(\gamma_*, 1)=q^{-N^1(\gamma_*)/6} \tag{B.6}$$

B.2. Proof of the Relation on the Free Energy

We shall now prove that the free energy $f_p(\beta H)$ satisfies (2.4.4). We first recall that the diluted partition functions of the original and dual models are related by

$$\mathbf{Z}^{\text{dil}}(V, \beta, p)=|B^p(V)|\, q^{-(p/d)N^p(V)}\, \mathbf{Z}^{\text{dil}}(V_*. \beta^*, d-p)$$

[this follows from (2.27), (2.4.3), and (2.2.5) for the complexes with trivial homology considered in the above relation]. Applying the relations (A.7), we obtain

$$|B^p(V)|=q^{\sum_{\alpha=0}^{p-1}(-1)^{\alpha+p-1}N^\alpha(V)}$$

We put

$$A^p(V)=\sum_{\alpha=0}^{p-1}(-1)^\alpha N^\alpha(V)+(-1)^p\frac{p}{d}N^p(V)$$

and show that $A^p(V)$ is a boundary term. To this end, we consider the expression

$$N^p(V)-C_d^jN^0(V)+C_d^jN^0(V)$$

Using an equivalent of the relation (B.3), we prove that $N^p(V)-C_d^jN^0(V)=B_j(V)$ is a negative boundary term for all j. Namely, $B_j(V)\leqslant 0$ $\forall j$ and

$$\frac{B_j(V)}{N_j(V)}\to 0 \qquad \text{if} \quad V\uparrow \mathbb{L}$$

Hence

$$A^p(V)=\sum_{j=0}^{p}(-1)^j B_j(V)+\left[\sum_{j=0}^{p}(-1)^jC_d^j+(-1)^{p+1}C_{d-1}^j\right]N^0(V)$$

From the identity

$$C_d^{j+1} - C_{d-1}^j = C_{d-1}^{j+1} \qquad \text{for each } j \text{ such that } 0 \leqslant j \leqslant d$$

we get

$$\sum_{j=0}^{p} (-1)^j C_d^j + (-1)^{p+1} C_{d-1}^j = 0 \qquad \text{for each } p$$

Thus, $A^p(V)$ is a boundary term, and the desired equality follows since $N^p(V) = N^{d-p}(V^*)$.

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